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Model Theory Homework 2

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Problem 6:
Show that the following conditions on $\mathbb{A}$ are equivalent.
(a) $\mathbb{A}$ is a model of the theory of the class of finite $\mathscr{L}$-structures.
(b) Every $\mathscr{L}$-sentence true in $\mathbb{A}$ holds in some finite finite $\mathscr{L}$-structure.
(c) $\mathbb{A}$ is elementarily equivalent to an ultraproduct of finite $\mathscr{L}$-structures.

Proof:
Let F be the class of all finite $\mathscr{L}$-structures.

$(a \to b) : \mathbb{A} \in F^{\perp\perp}$. Assume the contrary; there exists an $\mathscr{L}$-sentence ϕ such that $\mathbb{A} \models \phi$ and $\mathbb{M} \not\models \phi$ for all $\mathbb{M} \in F$. Having this, we get that $\mathbb{M} \models \neg\phi$ for all $\mathbb{M} \in F$ which implies that $\neg\phi \in F^{\perp}$. From later, we infer that $\mathbb{A} \notin F^{\perp\perp}$ which is a contradiction. Therefore, for all ϕ, where $\mathbb{A} \models \phi$ there exists $\mathbb{M} \in F$ such that $\mathbb{M} \models \phi$
$(b \to c)$: Let $I = \{\phi \in L-form : \mathbb{A} \models \phi\}$. Let $<$ be a relation on I defined as follow:
$r < s$ iff $\models s \to r$:
$(I, <)$ is a directed set since:
(1): If $r, s, t \in I$ such that $r < s$ and $s < t$ then we have that $\models s \to r$ and $\models t \to s$, therefore, $\models t \to s$ which means $r < s$ and we have that it is transitive.
(2) For $r, s \in I$, then $r \wedge s \in I$ and also $r, s < (r \wedge s)$.
Therefore, $(I, <)$ is a directed set.
Let $D = \{\Delta \subseteq I : \Delta = \{b : a < b\}; a \in I\}$. D is filter and let $\mathscr{U}$ be the ultra filter extended by D.
We claim that $\mathbb{A}$ is elementary equivalent to $\prod_{\phi\in I} \mathbb{M}_\phi/\mathscr{U}$ where $\mathbb{M}_\phi$ is chosen to be some $\mathbb{M}_\phi \in F$ where $\mathbb{M}_\phi \models \phi$ which exists by assuming (b):
Proof: It is enough to show that if $\mathscr{A} \models \phi$ then $\prod_{\phi\in I} \mathbb{M}_\phi/\mathscr{U} \models \phi$.
Let sentence ϕ such that $\mathbb{A} \models \phi$ then we have that $M_\phi \models \phi$ and also $\mathbb{M}_\psi \models \psi$ for all $\phi < \psi$.

Therefore, $\{\psi : \phi < \psi\} \subseteq \{\psi : \mathbb{M}_\psi \models \phi\}$. We also know that $\{\psi : \phi < \psi\} \in D \subseteq \mathscr{U}$. $\mathscr{U}$ is an ultrafilter, therefore, $\{\psi : \mathbb{M}_\psi \models \phi\} \in \mathscr{U}$ as desired.

$(c \to a)$: Assume $\mathbb{A}$ is elementary equivalent to an ultraproduct $\prod_{i \in I} \mathbb{B}_i / \mathscr{U}$ ($\mathbb{B}_i \in F$).
Now let ϕ be such that $\phi \in F^\perp$, therefore, $\mathbb{B} \models \phi$ for all $\mathbb{B} \in F$ and furthermore, $\{i \in I : \mathbb{B}_i \models \phi\} = I \in \mathscr{U}$. This would give us that $\mathbb{A} \models \phi$. $\phi \in F^\perp$ was arbitrary, therefore, $A \models \phi$ for all $\phi \in F^\perp$. This implies that $\mathbb{A} \in F^{\perp\perp}$ as desired.