Math 6000 Model Theory

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Homework 2

5. Look up the definitions of Σ_1^1 and Π_1^1 sentences if necessary, and include the definitions in your solution to the following problems.

(a) Show that ultraproducts preserve Σ_1^1-sentences.

(b) Give and example of a Π_1^1-sentence not preserved by ultraproducts.

A Σ_1^1 sentence is a second order sentence of the form

$$\exists R_1 \dots \exists R_k \varphi$$

where $R_1, \dots, R_k$ are second order variables and φ is a first order formula. A Π_1^1 sentence is a second order sentence of the form

$$\forall R_1 \dots \forall R_k \varphi$$

where $R_1, \dots, R_k$ are second order variables and φ is a first order formula.

Part a) Let ψ be a Σ_1^1 sentence $\exists R_1 \dots \exists R_k \varphi$ in the language $\mathcal{L}$. Then φ can be considered a first order formula in the extended language $\mathcal{L}' = \mathcal{L} \cup \{R_1, \dots, R_k\}$ where $R_1, \dots R_k$ are now considered as relations, not second order variables.

Let I be an infinite set and $\mathcal{U}$ an ultrafilter on I. Assume $\mathbb{A}_i$ are $\mathcal{L}$-structures such that $\mathbb{A}_i \models \psi$ for all $i \in I$. Extend each $\mathbb{A}_i$ to the $\mathcal{L}'$-structure $\mathbb{A}_i'$ so that $\mathbb{A}_i' \models \varphi$ for all i. By Łoś' Theorem, we have that $\prod_{\mathcal{U}} \mathbb{A}_i' \models \varphi$. Therefore $\prod_{\mathcal{U}} \mathbb{A}_i' \models \psi$. Note that $\prod_{\mathcal{U}} \mathbb{A}_i$ is an $\mathcal{L}$-structure and $\prod_{\mathcal{U}} \mathbb{A}_i'$ is an $\mathcal{L}'$-structure which was extended by $\{R_1, \dots, R_k\}$. Therefore, we have that $\prod_{\mathcal{U}} \mathbb{A}_i \models \psi$. Thus ultraproducts preserve Σ_1^1-sentences.

Part b) Ultraproducts do not preserve all Π_1^1 sentences. Consider the following example. Let $\mathcal{U}$ an ultrafilter on ω. Take $\mathbb{A} = \langle \mathbb{N}; \leq \rangle$. Let φ be the Π_1^1 sentence that says every nonempty subset of $\mathbb{N}$ has a least element, i.e.

$$\varphi : \forall S \exists x \forall y (x \in S \wedge (y \in S \rightarrow x \leq y)).$$

We will show that $\mathbb{A} \models \varphi$ but $\prod_{\mathcal{U}} \mathbb{A} \not\models \varphi$.

Clearly, $\mathbb{A} \models \varphi$. Define $\overline{a_n} \in \mathbb{N}^\omega$ as follows, $\overline{a_n} = (a_n^i)_{i\in\omega}$ where $a_n^i = 0$ if $i \leq n$ and $a_n^i = i - n$ for $i > n$. The collection of these $\overline{a_n}$ for $n \in \omega$ form a subset of $\mathbb{N}^\omega$. For any fixed k we have that $a_k^i = a_{k+1}^i$ for all $i \leq k$ and $a_k^i = a_{k+1}^i + 1$ for all $i > k$. Thus $a_k^i > a_{k+1}^i$ for infinitely many i's. Hence $\overline{a_{k+1}}$ is less than $\overline{a_k}$ in the ultraproduct. Therefore this subset of $\overline{a_n}$'s has no least element. Thus $\prod_{\mathcal{U}} \mathbb{A} \not\models \varphi$.