MODEL THEORY: HOMEWORK 2

KEVIN BERG, PAUL LESSARD, JEFF SHRINER

4. Two unrelated problems:

(a) Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. Show that an ultraproduct of ultraproducts of members of $\mathcal{K}$ is isomorphic to an ultraproduct of members of $\mathcal{K}$.

(b) Let $\mathcal{K}$ be a finite set of finite $\mathcal{L}$-structures. Show that any ultraproduct of members of $\mathcal{K}$ is isomorphic to some member of $\mathcal{K}$.

Proof.

(a) We follow a procedure suggested in [1, p. 15] – we will first construct a notion corresponding to a product of ultrafilters, and then produce the desired isomorphism. We also assume, without loss of generality, that $\mathcal{L}$ is a relational language in both parts (a) and (b) – this is possible since constants and n-ary functions can be thought of as unary and $n+1$-ary relations, respectively.

With this goal in mind, we begin by letting I denote a nonempty set and let $\{J_i\}_{i\in I}$ denote a system of nonempty sets. For each $i \in I$, let $\mathcal{U}_i$ denote an ultrafilter on J_i, and let $\mathcal{V}$ denote an ultrafilter on I. Let $K = \{(i,j) \ : \ i \in I, \, j \in J_i\}$ and define

$$\mathcal{W} = \{X \subseteq K \ : \ \{i \in I \ : \ \{j \in J_i \ : \ (i,j) \in X\} \in \mathcal{U}_i\} \in \mathcal{V}\}.$$

We claim that $\mathcal{W}$ is an ultrafilter on K. Since $J_i = \{j \in J_i \ : \ (i,j) \in K\} \in \mathcal{U}_i$ for each $i \in I$, it follows that $I = \{i \in I \ : \ J_i \in \mathcal{U}_i\} \in \mathcal{V}$, and thus $K \in \mathcal{W}$. It cannot be that $\emptyset \in \mathcal{W}$ since $\emptyset \notin \mathcal{U}_i$. The closure of $\mathcal{W}$ under finite intersection follows immediately from the closure of $\mathcal{U}_i$ and $\mathcal{V}$ under finite intersection, as does closure under supsets and the partitioning of complements. Our claim holds, and $\mathcal{W}$ will be used in the proof of the main result.

Let $\{\overline{A_{ij}}\}_{i\in I, j \in J_i}$ be an indexing of $\mathcal{L}$-structures[1] in $\mathcal{K}$, and let

$$\varphi : \prod_{\mathcal{V}} \left(\prod_{\mathcal{U}_i} \overline{A_{ij}} \right) \to \prod_{\mathcal{W}} \overline{A_{ij}}$$

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[1]The alternate notation $\overline{A} := (A, \sigma)$ is used to distinguish structures from ultrafilters given the frequency with which they will be appearing in close proximity for the duration of this proof.

be given by $\varphi([([t_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}}) = [(t_{ij})_{(i,j)\in K}]_{\mathcal{W}}$, where t_{ij} denotes the j-th coordinate of t_i. We claim that φ is our desired isomorphism – therefore, we will need to demonstrate that φ is a well-defined bijection, and that φ respects the relation symbols of $\mathcal{L}$.

We begin by showing that φ is well-defined and bijective. Suppose, then, that we have $[([t_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}} = [([t'_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}}$ for some t_i and t'_i. By construction, $\{i \in I \ : \ [t_i]_{\mathcal{U}_i} = [t'_i]_{\mathcal{U}_i}\} \in \mathcal{V}$, and this precisely means that $\{i \in I \ : \ \{j \in J_i \ : \ t_{ij} = t'_{ij}\} \in \mathcal{U}_i\} \in \mathcal{V}$. By definition, $\{(i,j) \in K \ : \ \{i \in I \ : \ \{j \in J_i \ : \ t_{ij} = t'_{ij}\} \in \mathcal{U}_i\} \in \mathcal{V}\} \in \mathcal{W}$, thus $\{(i,j) \in K \ : \ t_{ij} = t'_{ij}\} \in \mathcal{W}$, meaning precisely that $\varphi\left([([t_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}}\right) = \varphi\left([([t'_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}}\right)$, so φ is indeed well-defined. Similarly, f is injective: if $\varphi\left([([t_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}}\right) = \varphi\left([([t'_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}}\right)$, then $\{(i,j) \in K \ : \ t_{ij} = t'_{ij}\} \in \mathcal{W}$, so $\{(i,j) \in K \ : \ \{i \in I \ : \ \{j \in J_i \ : \ t_{ij} = t'_{ij}\} \in \mathcal{U}_i\} \in \mathcal{V}\} \in \mathcal{W}$, and it follows that $\{i \in I \ : \ \{j \in J_i \ : \ t_{ij} = t'_{ij}\} \in \mathcal{U}_i\} \in \mathcal{V}$ and $\{i \in I \ : \ [t_i]_{\mathcal{U}_i} = [t'_i]_{\mathcal{U}_i}\} \in \mathcal{V}$, so we conclude that $[([t_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}} = [([t'_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}}$. The surjectivity of φ is essentially immediate from definitions – if $x \in \prod_{\mathcal{W}} \overline{A_{ij}}$, then $x = [(t_{ij})_{(i,j)\in K}]_{\mathcal{W}} = \varphi\left([([t_i]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}}\right)$.

It remains to be shown that φ respects the relation symbols of $\mathcal{L}$ – therefore, we let R be an n-ary relation symbol of $\mathcal{L}$ and check that, given $x_1, \ldots, x_n \in \prod_{\mathcal{V}}\left(\prod_{\mathcal{U}_i} \overline{A_{ij}}\right)$, $(x_1, \ldots, x_n) \in R^{\prod_{\mathcal{V}}\left(\prod_{\mathcal{U}_i} \overline{A_{ij}}\right)}$ if and only if $(\varphi(x_1), \ldots, \varphi(x_n)) \in R^{\prod_{\mathcal{W}} \overline{A_{ij}}}$. For each x_k, we write $x_k = [([t_i^k]_{\mathcal{U}_i})_{i\in I}]_{\mathcal{V}}$ for notational purposes. Using the definitions of the interpretation of relation symbols under products and ultraproducts, we arrive at the following chain of implications:

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\begin{aligned}
(x_1, \ldots, x_n) \in R^{\prod_{\mathcal{V}}\left(\prod_{\mathcal{U}_i} \overline{A_{ij}}\right)} &\text{ if and only if } \left\{ i \in I \ : \ \left([t_i^1]_{\mathcal{U}_i}, \ldots, [t_i^n]_{\mathcal{U}_i}\right) \in R^{\prod_{i\in I}\left(\prod_{\mathcal{U}_i} \overline{A_{ij}}\right)} \right\} \in \mathcal{V} \\
&\text{ if and only if } \left\{ i \in I \ : \ \left\{ j \in J_i \ : \ \left(t_{ij}^1, \ldots, t_{ij}^n\right) \in R^{\overline{A_{ij}}} \right\} \in \mathcal{U}_i \right\} \in \mathcal{V} \\
&\text{ if and only if } \left\{ (i,j) \in K \ : \ \left(t_{ij}^1, \ldots, t_{ij}^n\right) \in R^{\overline{A_{ij}}} \right\} \in \mathcal{W} \\
&\text{ if and only if } (\varphi(x_1), \ldots, \varphi(x_n)) \in R^{\prod_{\mathcal{W}} \overline{A_{ij}}}.
\end{aligned}
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Consequently, we indeed have that given $x_1, \ldots, x_n \in \prod_{\mathcal{V}}\left(\prod_{\mathcal{U}_i} A_{ij}\right)$, $(x_1, \ldots, x_n) \in R^{\prod_{\mathcal{V}}\left(\prod_{\mathcal{U}_i} \overline{A_{ij}}\right)}$ if and only if $(\varphi(x_1), \ldots, \varphi(x_n)) \in R^{\prod_{\mathcal{W}} \overline{A_{ij}}}$, and so φ respects the relation symbols of $\mathcal{L}$.

Since φ is a bijection and respects the relation symbols of our purely relational $\mathcal{L}$, we have produced an isomorphism between an ultraproduct of ultraproducts of members of $\mathcal{K}$ and an ultraproduct of members of $\mathcal{K}$.

(b) Let $\mathcal{K} = \{\mathbb{A}_1, \dots, \mathbb{A}_n\}$, I a nonempty index set, and $\prod_{\mathcal{U}} \mathbb{A}_{j_i}$ ($j_i \in \{1, \dots, n\}$) an ultraproduct over some ultrafilter $\mathcal{U}$ of I. Note that we may assume that $|I|$ is infinite, for if $|I|$ were finite, then $\mathcal{U}$ is principal and the result follows.

Now let $J_k = \{i \in I \mid \mathbb{A}_{j_i} = \mathbb{A}_k\}$, where $1 \leq k \leq n$.

Claim 1. There exists an $\ell \in \{1, \dots, n\}$ such that $J_\ell \in \mathcal{U}$ and $J_k \notin \mathcal{U}$ for $k \neq \ell$.

Proof of Claim 1. Suppose towards a contradiction that $J_k \notin \mathcal{U}$ for all $k \in \{1, \dots, n\}$. Then $(I \setminus J_k) \in \mathcal{U}$ for all k, implying $\cap_{k=1}^n (I \setminus J_k) = I \setminus (\cup_{k=1}^n J_k) = \emptyset \in \mathcal{U}$, a contradiction. Thus there exists an ℓ such that $J_\ell \in \mathcal{U}$. Then since $\mathcal{U}$ is closed under finite intersections and $J_\ell \cap J_k = \emptyset$ for all $k \neq \ell$, it follows that $J_k \notin \mathcal{U}$ for all $k \neq \ell$.

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Fix $\mathbb{A}_\ell = \{b_1, \dots, b_m\}$ from Claim 1, and let $(a_i)_{i\in I} \in \prod \mathbb{A}_{j_i}$. We define $L_k = \{i \in J_\ell \mid a_i = b_k\}$, where $1 \leq k \leq m$.

Claim 2. For each $(a_i)_{i\in I} \in \prod \mathbb{A}_{j_i}$, there exists an $s \in \{1, \dots, m\}$ such that $L_s \in \mathcal{U}$ and $L_k \notin \mathcal{U}$ for $k \neq s$. Further, if $[(a_i)_{i\in I}]_{\theta_\mathcal{U}} = [(b_i)_{i\in I}]_{\theta_\mathcal{U}}$, $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ correspond to the same s.

Proof of Claim 2. Suppose towards a contradiction that $L_k \notin \mathcal{U}$ for all $k \in \{1, \dots, m\}$. Then $I \setminus L_k \in \mathcal{U}$ for all k, implying $\cap_{k=1}^m (I \setminus L_k) = I \setminus (\cup_{k=1}^m L_k) = I \setminus J_\ell \in \mathcal{U}$, a contradiction. Thus there is an s such that $L_s \in \mathcal{U}$, and since $L_s \cap L_k = \emptyset$ for all $k \neq s$, we have that $L_k \notin \mathcal{U}$ for all $k \neq s$. From Claim 1 and the first part of Claim 2, we see that every tuple $(a_i)_{i\in I}$ has exactly one element (which must be from $\mathbb{A}_\ell$) which occurs in almost every coordinate. Thus, if $[(a_i)_{i\in I}]_{\theta_\mathcal{U}} = [(b_i)_{i\in I}]_{\theta_\mathcal{U}}$, $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ correspond to the same s.

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Using the notation established in Claims 1 and 2, we define the map

$$\underset{[(a_i)_{i\in I}]_{\theta_\mathcal{U}} \mapsto b_s}{\varphi : \prod_\mathcal{U} \mathbb{A}_{j_i} \to \mathbb{A}_\ell}$$

and claim that φ is the desired isomorphism. By Claim 2 φ is well-defined. If $b_k \in \mathbb{A}_\ell$, then the equivalence class of $(a_i)_{i\in I}$, $a_i = b_k$ for all $i \in J_\ell$, is mapped to b_k under φ, so φ is onto. Finally, if $\varphi([(a_i)_{i\in I}]_{\theta_\mathcal{U}}) = \varphi([(c_i)_{i\in I}]_{\theta_\mathcal{U}}) = b_k$, then b_k occurs almost everywhere in $(a_i)_{i\in I}$ and $(c_i)_{i\in I}$, so $[(a_i)_{i\in I}]_{\theta_\mathcal{U}} = [(c_i)_{i\in I}]_{\theta_\mathcal{U}}$, and φ is one-to-one.

Since functions and constants may be expressed as relations, to prove that φ is an embedding it suffices to show that φ preserves relations. To that end, let R be k-ary relation symbol, and $[a_1]_{\theta_\mathcal{U}}, \dots, [a_k]_{\theta_\mathcal{U}} \in \prod_\mathcal{U} \mathbb{A}_{j_i}$. Then $\varphi([a_i]_{\theta_\mathcal{U}}) = b_{j_i}$ implies b_{j_i} is (the only element) in almost every coordinate of a_i, so that

$$\begin{aligned}
(\varphi([a_1]_{\theta_\mathcal{U}}), \ldots, \varphi([a_k]_{\theta_\mathcal{U}})) \in R^{\mathbb{A}_\ell} &\leftrightarrow (b_{j_1}, \ldots, b_{j_k}) \in R^{\mathbb{A}_\ell} \\
&\leftrightarrow [\![R^{\mathbb{A}_i}((a_1)_i, \ldots, (a_k)_i)]\!] \in \mathcal{U} \\
&\leftrightarrow ([a_1]_{\theta_\mathcal{U}}, \ldots, [a_k]_{\theta_\mathcal{U}}) \in R^{\prod_\mathcal{U} \mathbb{A}_{j_i}}
\end{aligned}$$

□

References

[1] J. Donald Monk, *Lecture Notes on Model Theory: 1. Structures*, `http://euclid.colorado.edu/~monkd/m6000/m6000_01.pdf`, 2012.