MODEL THEORY: HOMEWORK 1

ALI LOTFI , PETER TANKSALVALA

3: Show that any structure is embeddable in an ultraproduct of its finitely generated substructures. Conclude that any universal class is generated by its finitely generated members. Show that this statement about universal classes is not true for arbitrary elementary classes.

Claim 1: Any structure is embeddable in an ultraproduct of its finitely generated substructures.
Proof:
Case 1: $\mathbb{A}$ is finitely generated.
Any structure $\mathbb{A}$ is embeddable into an ultrapower of itself. An ultrapower is an ultraproduct, so A is embeddable into an ultraproduct of finitely generated substructures.

Case 2: $\mathbb{A}$ is infinitely generated:
Let I denote the set of all nonempty finite subsets of A the set underlying $\mathbb{A}$. Define $\mathbb{M}_i$ to be the substructure of $\mathbb{A}$ generated by $i \in I$. Let J_a denote the set of all $i \in I$ such that $a \in \mathbb{M}_i$.

Define $F := \{J_a \in \mathcal{P}(I) | a \in A\}$. A subset i of A is in J_a iff it is finite and $a \in \mathbb{M}_i$. Recognize that a single element is always finitely generated, so no J_a is empty, meaning the subset F of $\mathcal{P}(I)$ is proper.

Let i_j be some element of J_{a_j} for all $j \in \omega$. Consider $\bigcap\limits_{j=1}^{n} J_{a_j}$. $\bigcup\limits_{j=1}^{n} i_j$ is a finite union of finite sets, so is finite. Moreover, all the a_j are elements of $\mathbb{M}_{\bigcup_{j=1}^{n} i_j}$, because $a_j \in \mathbb{M}_{i_j}$ is contained in $\mathbb{M}_{\bigcup_{j=1}^{n} i_j}$ for every $j \in \omega$. This means that $\mathbb{M}_{\bigcup_{j=1}^{n} i_j}$ has a finite set of generators with every a_j as elements, so $\bigcup\limits_{j=1}^{n} i_j$ is an element of $\bigcap\limits_{j=1}^{n} J_{a_j}$ meaning it is nonempty. Thus F has the finite intersection property, so generates a proper filter meaning an ultrafilter $\mathcal{U}$ exists containing the filter $\langle F \rangle$ generated by F.

For each element $i \in I$, choose an element of i. Define the elements $f_a \in \Pi_{\mathcal{U}} \mathbb{M}_i$, such that $f_a(i) = a$ for all i such that $a \in \mathbb{M}_i$ and the chosen element of i whenever $a \notin \mathbb{M}_i$. Define a function φ' from $\mathbb{A}$ to $\Pi \mathbb{M}_i$ by $\varphi(a) = f_a$ and φ to be the composition of φ' and the projection map from

$\Pi\mathbb{M}_i$ to $\Pi_{\mathcal{U}}\mathbb{M}_i$. Assume $a_1 \neq a_2$ are elements of $\mathbb{A}$. For every $j \in I$ such that a_1 and a_2 are elements of $\mathbb{M}_j$, $f_{a_1}(j) \neq f_{a_2}(j)$. Realize the set of all such j is the intersection of two elements of $\mathcal{U}$, namely J_{a_1} and J_{a_2}. Thus f_{a_1} and f_{a_2} can agree only at a subset of the complement of an element of the ultrafilter, so $[f_{a_1}] \neq [f_{a_2}]$. Thus φ is injective.

For arbitrary relation R in the language of $\mathbb{A}$, let $(a_1, ..., a_n) \in R^{\mathbb{A}}$. $([f_{a_1}], ..., [f_{a_n}]) \in R^{\Pi_{\mathcal{U}}\mathbb{M}_i}$ iff $\{i | (f_{a_1}(i), ..., f_{a_n}(i)) \in R^{\mathbb{M}_i}\} \in \mathcal{U}$. Recognize that $a_1, ..., a_n \in \mathbb{M}_i$ implies $(f_{a_1}(i), ..., f_{a_n}(i)) \in R^{\mathbb{M}_i}$. This means $\{i | \bigwedge_{j=1}^{n}(a_j \in \mathbb{M}_i)\} \subseteq \{i | (f_{a_1}(i), ..., f_{a_n}(i)) \in R^{\mathbb{M}_i}\}$. Recognize $\{i | \bigwedge_{j=1}^{n}(a_j \in \mathbb{M}_i)\} = \bigcap_{j=1}^{n} J_{a_j}$, so $\{i | \bigwedge_{j=1}^{n}(a_j \in \mathbb{M}_i)\}$ is the finite intersection of elements of $\mathcal{U}$, thus is an element of $\mathcal{U}$. This implies $\{i | (f_{a_1}(i), ..., f_{a_n}(i)) \in R^{\mathbb{M}_i}\} \in \mathcal{U}$, so φ preserves relation.

The case for arbitrary functions follows similarly and constants clearly follow from the definition of φ.

So $\mathbb{A}$ is embeddable in an ultraproduct of its finitely generated subsets.

Claim 2: Any universal class is generated by its finitely generated elements.

Proof:

That is to say, for any universal class $\mathcal{K}$ axiomatized by Σ and class $\mathcal{O}$ of finitely generated elements of $\mathcal{K}$, show $(\mathcal{O}^{\perp})^{\perp} = \mathcal{K}$.

To show $(\mathcal{O}^{\perp})^{\perp}$ is contained in $\mathcal{K}$:

Recall that $\mathcal{O} \subseteq \mathcal{K}$, so $(\mathcal{O}^{\perp})^{\perp} \subseteq (\mathcal{K}^{\perp})^{\perp}$. $\mathcal{K}$ is a universal class, so $(\mathcal{K}^{\perp})^{\perp} = \mathcal{K}$ meaning $(\mathcal{O}^{\perp})^{\perp} \subseteq \mathcal{K}$.

To show $\mathcal{K}$ is contained in $(\mathcal{O}^{\perp})^{\perp}$:

Σ is universally quantified, so $\mathbf{A} \models \Sigma$ implies $\mathbf{M}_i \models \Sigma$ for $\mathbf{M}_i$ a substructure of $\mathbf{A}$. This implies every finitely generated substructure of $\mathbf{A}$ is an element of $\mathcal{K}$ so is an element of $\mathcal{O}$. Moreover, $(\mathcal{O}^{\perp})^{\perp}$ is closed under ultraproducts, so any ultraproduct of such $\mathbf{M}_i \in \mathcal{O} \subseteq (\mathcal{O}^{\perp})^{\perp}$ is an element of $(\mathcal{O}^{\perp})^{\perp}$.

Recall that $\mathbf{A}$ is embeddable in an ultraproduct of its finitely generated substructures $\Pi_{\mathcal{U}}\mathbf{M}_i$. This means that $\Pi_{\mathcal{U}}\mathbf{M}_i \models \mathcal{O}^{\perp}$ implies $\mathbf{A} \models \mathcal{O}^{\perp}$, so $\mathbf{A} \in (\mathcal{O}^{\perp})^{\perp}$.

$(\mathcal{O}^{\perp})^{\perp} = \mathcal{K}$ implies that $\mathcal{O}$ generates $\mathcal{K}$ as needed.

Part 3: Show that this statement about universal classes is not true for

arbitrary elementary classes.
A structure $\mathbf{A}$ is generated by A' if the set underlying $\mathbf{A}$ is the smallest set containing A' closed under the operations of $\mathbf{A}$.

Use the language $\mathcal{L}$ of posets, namely the symbol $\geq$ with no constants or functions. Lacking any functions, an $\mathcal{L}-$structure generated by A' is the $\mathcal{L}-$structure with underlying set A'. This means that a finitely generated $\mathcal{L}-$structure is finite.

Let $\mathcal{K}$ be the elementary class of dense linearly ordered sets without endpoints. No finite dense linearly ordered sets without endpoints exists, so no finitely generated elements of $\mathcal{K}$ exist. This means that the subclass of finitely generated elements is empty. $(\exists x(x \neq x)) \in \emptyset^{\perp}$, but is not an element of $\mathcal{K}^{\perp}$. This means that $\mathcal{K}$ isn't generated by its finitely generated structures.