2. Let $G = \langle \mathbb{Z}, R\rangle$ be the structure with a binary relation $R = \{(x, y) : x = y-1 \text{ or } x = y+1\}$. Let $2G = \langle \mathbb{Z} \sqcup \mathbb{Z}, R'\rangle$ be the structure with a disjoint union of two copies of the integers—that is, $(\mathbb{Z}\times\{a\})\cup(\mathbb{Z}\times\{b\})$ with $a, b \notin \mathbb{Z}$—and the binary relation $R' = \{((x_1, x_2), (y_1, y_2)) : R(x_1, y_1) \text{ and } x_2 = y_2\}$, with $R$ defined as above. Let $I$ be an infinite indexing set so that, by Theorem 2 from the Feb 15 class notes, we can define $\mathcal{U}$ to be a regular ultrafilter on $I$. Since both $G$ and $2G$ are infinite sets, the Frayne-Morel-Scott Theorem allows us to calculate the size of the ultrapowers on these sets: $|\prod_{\mathcal{U}} G| = |G|^{|I|} = |2G|^{|I|} = |\prod_{\mathcal{U}} 2G|$. We now show that these like-sized ultrapowers have isomorphic structure.

   We view $G$ and $2G$ as graphs where the vertices represent elements and segments between vertices exist iff the corresponding elements are related by $R$ or $R'$. Then both $G$ and $2G$ model the following first-order properties: (1) they're infinite, (2) every vertex is connected to exactly two other vertices; and (3) they have no finite cycles of size greater than 2. The first two are clearly first-order properties. To demonstrate the third is a first-order property, we can formulate it as the following first-order statements for each $n > 2$, which essentially states that for each collection of $n$ distinct vertices, there is a vertex that is connected to at most one of the other vertices in the collection: $\forall x_1, ..., x_n(\bigwedge_{1\le i<j\le n}(x_i \neq x_j) \to \exists x_a \forall x_b \forall x_c((\bigvee_{1\le i\le n} x_a = x_i) \wedge ((x_b \neq x_c \wedge (\bigvee_{1\le i\le n} x_b = x_i) \wedge (\bigvee_{1\le i\le n} x_c = x_i)) \to (R(x_a, x_b) \to \neg R(x_c, x_a)))))$. Any cycle of length $n > 2$ would violate the corresponding first-order sentence because each vertex would be connected to more than one of the other vertices. Also, if a collection of $n$ vertices has no cycles of length greater than 2, then one could move from each vertex to a new vertex indefinitely and, since there are only finitely many vertices and no cycles, one would eventually hit a vertex that is only connected to one other vertex in the collection.

   By Łos's Theorem, ultrapowers preserve first-order sentences, so each of the aforementioned ultrapowers satisfy these sentences. Note that a structure of uncountable size $\kappa$ with no finite cycles where each element is related to exactly two other elements must be a very specific type of structure; namely, $\kappa$ disjoint union of $G$'s. Informally, this becomes clear if we: (1) put an element into a set, (2) add its two related elements into the set, (3) continue with the next two related elements and repeat a countably many times, and (4) repeat this process for any element left out of all previously built sets. The process builds the connected components, each of which are isomorphic to $G$. Since the ultra-powers defined above are disjoint unions of the same cardinal of $G$'s, they are isomorphic.