MODEL THEORY: HOMEWORK 2

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1. An ultrafilter is *uniform* if all of its sets have the same size. Show that a regular ultrafilter is uniform.

Proof. Let $\mathcal{U}$ be an ultrafilter on a set I, and $E \subseteq \mathcal{U}$ be such that $|E| = |I|$ and every $i \in I$ is a member of finitely many sets in E.

Assume towards a contradiction that there is an $A \in \mathcal{U}$ such that $|A| < |I|$. Then since $|E| = |I|$, we may index the members of E by the set I. Then we note that for each $a \in A$, the set $X_a = \{i \in I : a \in E_i\}$ is finite so that $\cup_{a \in A} X_a \neq I$ by a cardinality argument. But this implies that there exists $j \in I$ such that $a \notin E_j$ for all $a \in A$, so that $E_j \cap A = \emptyset$. This is a contradiction, as $E_j, A \in \mathcal{U}$ and $\mathcal{U}$ is closed under finite intersection. Thus we must have that $|A| = |I|$ for all $A \in \mathcal{U}$.

$\square$

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