MODEL THEORY: HOMEWORK 1

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7. If $\mathbf{A}$ is a structure, then a subset $U \subseteq A$ is definable if there is some formula $\varphi(x)$ such that $u \in U$ iff $\mathbf{A} \models \varphi[u]$. Explain why the following sets are definable in $\mathbb{R} = \langle \{\mathbf{reals}\}; \cdot, + \rangle$.

(a) The unit interval.
(b) $\{n\}$ for any integer n.
(c) $\{a\}$ for any real algebraic number a.

Proof. First, we define several formulas which will be of use:

$$\begin{aligned}\varphi_0(x) &= \forall y((x + y = y) \wedge (y + x = y))\\ \varphi_1(x) &= \forall z((x \cdot z = z) \wedge (z \cdot x = z))\\ \varphi_{\leq}(x, y) &= \exists z(x + z \cdot z = y)\end{aligned}$$

(a) Consider the formula

$$\varphi_{[0,1]}(x) = \exists y(\varphi_0(y) \wedge \varphi_{\leq}(y, x)) \wedge \exists w(\varphi_1(w) \wedge \varphi_{\leq}(x, w))$$

Note that since $\mathbb{R}$ contains the multiplicative identity 1, $\mathbb{R} \models \varphi_{[0,1]}[u]$ iff u is non-negative and less than or equal to 1, as desired.

(b) Fix an integer n. If n is positive, we define $\{n\}$ by

$$\varphi_n(x) = \exists w(\varphi_1(w) \wedge (x = \overbrace{w + \cdots + w}^{n \text{ times}}))$$

If n is negative, we define $\{n\}$ by

$$\varphi_n(x) = \exists w \exists m(\varphi_1(w) \wedge \varphi_0(m + w) \wedge (x = \overbrace{m + \cdots + m}^{-n \text{ times}}))$$

Finally, if $n = 0$, we define $\{n\}$ by $\varphi_0(x)$.

(c) First, note that every rational number is defined in $\mathbb{R}$ similarly to how we defined integers in part (b). For example, if m and n are positive, then $\frac{n}{m}$ is defined by

$$\varphi_{\frac{n}{m}}(x) = \exists w(\varphi_1(w) \wedge (\overbrace{x + \cdots + x}^{m \text{ times}} = \overbrace{w + \cdots + w}^{n \text{ times}}))$$

Date: February 5, 2016.

Now fix a real algebraic number a. Let $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$, $c_i \in \mathbb{Q}$ for $0 \leq i \leq n-1$, be the minimal polynomial of a, and let (s, r) be an open interval with $s, r \in \mathbb{Q}$ such that (s, r) contains a but no other root of $p(x)$. Then we define $\{a\}$ by the formula

$$\begin{aligned} \varphi_a(x) = & \exists y_0 \cdots \exists y_{n-1} \left(\bigwedge_{i=0}^{n-1} (\varphi_{c_i}(y_i)) \wedge \varphi_0(x^n + y_{n-1}x^{n-1} + \cdots + y_1 x + y_0) \right) \\ & \wedge \exists z(\varphi_s(z) \wedge \varphi_{\leq}(z, x)) \wedge \exists y(\varphi_r(y) \wedge \varphi_{\leq}(x, y)). \end{aligned}$$

$\square$