MODEL THEORY: HOMEWORK 1

JEFF SHRINER, ATHENA SPARKS, PETER TANKSALVALA

6: Show that the class, $\mathcal{K}$, of well ordered sets is not elementary. (Use the language of partially ordered sets.)

Proof:

Let $\Sigma = \mathcal{K}^{\perp}$. Define the constants $c_0,\ c_1,\ c_2, ...$, the relation $\geq$. Let $\mathcal{L}$ be the language of such constants and $\geq$.

Define the sets $M_n = \{x_0, x_1, ..., x_n\}$ for each $n \in \omega$ to be a set of $n+1$ elements. Define each $\mathcal{L}$-structure $\mathbf{M}_n$ over M_n by, for $i \in \{0, 1, ..., n-1\}$, $c_i^{\mathbf{M}_n} = x_i$ and $c_i^{\mathbf{M}_n} = x_n$ for all $i \geq n$ with $c_i \neq c_j$ for $i \neq j$ and $i, j \in \{0, 1, ..., n\}$. Each $\mathbf{M}_n$ is a totally ordered finite set, thus is well ordered.

Note that, for arbitrary $n \in \omega$, $\Sigma \cup \{(c_i \geq c_{i+1}) \wedge (c_i \neq c_{i+1}) | i \in I\}$ is satisfied by $\mathbf{M}_n$ for every $I \subseteq \{0,\ 1, ...,\ n-1\}$, which implies every finite subset of $\Sigma \cup \{(c_i \geq c_{i+1}) \wedge (c_i \neq c_{i+1}) | i \in \omega\}$ is satisfiable. By the compactness theorem, this implies $\Sigma \cup \{(c_i \geq c_{i+1}) \wedge (c_i \neq c_{i+1}) | i \in \omega\}$ is satisfiable.

Let $\mathbf{N}$ be an $\mathcal{L}-$structure satisfying $\Sigma \cup \{(c_i \geq c_{i+1}) \wedge (c_i \neq c_{i+1}) | i \in \omega\}$ existing by the compactness theorem. Note that $\mathbf{N}$ contains an infinite descending chain, so is not a well ordered set, and $\mathbf{N} \models \Sigma \cup \{(c_i \geq c_{i+1}) \wedge (c_i \neq c_{i+1}) | i \in \omega\}$ implies $\mathbf{N} \models \Sigma$. This means $(\mathcal{K}^{\perp})^{\perp} = \Sigma^{\perp} \neq \mathcal{K}$, making $\mathcal{K}$ not an elementary class.