4. We wish to show that the class of simple groups, $\mathcal{K}$, is not elementary in the language of groups, $\mathcal{L}$. We suppose that $\mathcal{K}$ is elementary in anticipation of a contradiction – therefore, let $T$ be an $\mathcal{L}$-theory such that $\mathcal{K} = \{\mathcal{M} \ : \ \mathcal{M} \vDash T\}$. We use the shorthand notation $\phi_n(x)$ to denote the $\mathcal{L}$-formula $x \cdot x \cdot (n\text{-times}) \cdot x = e$. Let $T' = T \cup \{\forall x \forall y \ x \cdot y = y \cdot x\} \cup \{\forall x \ \neg(x = e) \to \neg\phi_n(x) \ : \ n \in \mathbb{Z}_{\geq 2}\}$.

   We claim that any finite subset of $T'$ has a simple group as a model. Let $S$ be such a finite subset. If $S \subset T \cup \{\forall x \forall y \ x \cdot y = y \cdot x\}$, then $\mathbb{Z}/2\mathbb{Z} \vDash S$, and the claim holds. If $S \not\subset T \cup \{\forall x \forall y \ x \cdot y = y \cdot x\}$, then by finiteness there exists a maximal integer $N$ such that the sentence $\forall x \ \neg(x = e) \to \neg\phi_N(x)$ is in $S$. Let $p$ denote a prime with $p > N$ – then $\mathbb{Z}/p\mathbb{Z} \vDash S$. In either case, the claim holds.

   By compactness, therefore, we have that $T'$ is satisfiable – however, $T'$ describes an abelian simple group where every non-identity element has infinite order, and the only abelian simple groups are the cyclic groups of prime order [1, p. 29]. This is a contradiction, and thus $\mathcal{K}$ is not elementary.

# References

[1] I. M. Isaacs, *Finite Group Theory*, American Mathematical Society, Providence, Rhode Island, 2008.