Math 6000 Model Theory

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Homework 1

3. Show that the following pairs of abelian groups are not elementarily equivalent.
   (a) $\mathbb{Z}$ and $\mathbb{Q}$
   (b) $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$

To show that the above pairs of abelian groups are not elementarily equivalent we will find a first order sentence σ that is true in one structure but false in the other.

Part a) Let σ be the sentence $\forall x \exists y (x = y + y)$. If $x \in \mathbb{Q}$, then $\frac{x}{2} \in \mathbb{Q}$ and $\frac{x}{2} + \frac{x}{2} = x$. Thus $\mathbb{Q} \models \sigma$. Since $x + x$ is even for any integer x, no odd integer can equal $x + x$ for any $x \in \mathbb{Z}$. Thus, $\mathbb{Z} \not\models \sigma$.

Part b) Now let σ be the sentence $\exists x \forall y \exists z((y = x+z+z) \vee (y = z+z))$. $\mathbb{Z} \models \sigma$ since if we choose $x = 1$, then σ is equivalent to the statement every integer is odd or even. We will show that $\mathbb{Z} \times \mathbb{Z} \not\models \sigma$.

Assume toward the contrary that $\mathbb{Z} \times \mathbb{Z} \models \sigma$. Then there exists an element $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that for all $(c, d) \in \mathbb{Z} \times \mathbb{Z}$ there exists $(e, f) \in \mathbb{Z} \times \mathbb{Z}$ so that $(c, d) = (a, b) + (e, f) + (e, f)$ or $(c, d) = (e, f) + (e, f)$. In particular, there exists $(e, f) \in \mathbb{Z} \times \mathbb{Z}$ such that $(1, 0) = (a, b) + (e, f) + (e, f)$ or $(1, 0) = (e, f) + (e, f)$. Since $1 \neq e + e$ for any $e \in \mathbb{Z}$, we have that $(1, 0) = (a, b) + (e, f) + (e, f)$. Therefore, $1 = a + e + e$ and hence a is odd. We also have that there exists $(g, h) \in \mathbb{Z} \times \mathbb{Z}$ such that $(0, 1) = (a, b) + (g, h) + (g, h)$ or $(0, 1) = (g, h) + (g, h)$. Since $1 \neq h + h$ for any $h \in \mathbb{Z}$, we have $(0, 1) = (a, b) + (g, h) + (g, h)$. Therefore, $0 = a + g + g$ and hence a is even. Since a cannot be both even and odd, we have our contradiction. Thus $\mathbb{Z} \times \mathbb{Z} \not\models \sigma$.