Let $\mathscr{L}$ be a language. Show that if $\alpha : \mathbf{A} \longrightarrow \mathbf{B}$ is an isomorphism of $\mathscr{L}$-structures, then for all $\mathscr{L}$-formulae φ and for all $\mathbf{A}$-valuations v, we have that $\mathbf{A} \models \varphi[v]$ if and only if $\mathbf{B} \models \varphi[\alpha(v)]$.

We prove this claim by way of the following lemma.

Lemma. *For all $\mathscr{L}$-terms t, and all valuations v valued in A it follows that*

$$\alpha\left(t^{\mathbf{A}}[v]\right) = t^{\mathbf{B}}[\alpha(v)]$$

.

Proof. The proof proceeds by induction on the set of valuations of $\mathscr{L}$-terms:

constants: as α is an isomorphism of structures, $\alpha\left(c^{\mathbf{A}}\right) = c^{\mathbf{B}}$ for all constant symbols c;

variables: given variable $\overline{x} = (x_1, \dots)$ and a valuation $v : x_i \mapsto a_i$ we have that

$$\begin{aligned} \alpha(\overline{x}[v]) &= \alpha((a_1, \dots)) \\ &= (\alpha(a_1), \dots) \\ &= \overline{x}[\alpha(v)]; \end{aligned}$$

and

composition: if $t_1, \dots, t_n$ are $\mathscr{L}$-terms such that $\alpha\left(t_i^{\mathbf{A}}\right) = t_i^{\mathbf{B}}$ and f is a function symbol, then

$$\begin{aligned} \alpha\left(f^{\mathbf{A}}\left(t_1^{\mathbf{A}}, \dots, t_n^{\mathbf{A}}\right)\right) &= f^{\mathbf{B}}\left(\alpha\left(t_1^{\mathbf{B}}, \dots, t_n^{\mathbf{B}}\right)\right) \\ &= f^{\mathbf{B}}\left(t_1^{\mathbf{B}}, \dots, t_n^{\mathbf{B}}\right). \end{aligned}$$

□

We may now attend to the proof of the problem.

Proof. The proof proceeds by induction of the set of $\mathscr{L}$-formulae. Suppose v to be a valuation landing in $\mathbf{A}$.

atomic formulae: Suppose φ to be $t_1 = t_2$, then for a valuation v we have that

$$\begin{aligned} \mathbf{A} \models \varphi[v] &\leftrightarrow t_1^{\mathbf{A}}[v] = t_2^{\mathbf{A}}[v] \\ &\leftrightarrow \alpha\left(t_1^{\mathbf{A}}[v]\right) = \alpha\left(t_2^{\mathbf{A}}[v]\right) \\ &\leftrightarrow \alpha\left(t_1^{\mathbf{A}}\right)[\alpha(v)] = \alpha\left(t_2^{\mathbf{A}}\right)[\alpha(v)] \\ &\leftrightarrow t_1^{\mathbf{B}}[\alpha(v)] = t_2^{\alpha}[\alpha(v)] \\ &\leftrightarrow B \models \varphi[\alpha(v)]. \end{aligned}$$

Similarly if φ is $R(t_1, \dots, t_n)$ then

$$\begin{aligned} \mathbf{A} \models \varphi[v] &\leftrightarrow \left(t_1^{\mathbf{A}}, \dots, t_n^{\mathbf{A}}\right)[v] \in R^{\mathbf{A}} \\ &\leftrightarrow \left(t_1^{\mathbf{A}}[v], \dots, t_n^{\mathbf{A}}[v]\right) \in R^{\mathbf{A}} \\ &\leftrightarrow \left(\alpha\left(t_1^{\mathbf{A}}[v]\right), \dots, \alpha\left(t_1^{\mathbf{A}}[v]\right)\right) \in \alpha\left(R^{\mathbf{A}}\right) \\ &\leftrightarrow \left(t_1^{\mathbf{B}}[\alpha(v)], \dots t_1^{B}[\alpha(v)]\right) \in R^{\mathbf{B}} \\ &\leftrightarrow B \models \varphi[\alpha(v)]. \end{aligned}$$

negation: Suppose φ is $\neg\psi$ where the claim has been proved for ψ, then

$$\begin{aligned}\mathbf{A} \models \varphi[v] &\leftrightarrow \mathbf{A} \not\models \psi[v] \\ &\leftrightarrow \mathbf{B} \not\models \psi[\alpha(v)] \\ &\leftrightarrow \mathbf{B} \models \varphi[\alpha(v)].\end{aligned}$$

conjunction: Suppose φ to be $\psi \wedge \vartheta$, then by induction,

$$\begin{aligned}\mathbf{A} \models \varphi[v] &\leftrightarrow \mathbf{A} \models \psi[v] \wedge \vartheta[v] \\ &\leftrightarrow \mathbf{A} \models \psi[v] \wedge \mathbf{A} \models \vartheta[v] \\ &\leftrightarrow \mathbf{B} \models \psi[\alpha(v)] \wedge \mathbf{B} \models \psi[\alpha(v)] \\ &\leftrightarrow \mathbf{B} \models \psi[\alpha(v)] \wedge \psi[\alpha(v)] \\ &\leftrightarrow \mathbf{B} \models \varphi[\alpha(v)].\end{aligned}$$

existential quantification: Suppose φ to be $\exists v_1(\psi(v_1))$, then

$$\begin{aligned}\mathbf{A} \models \varphi[v] &\leftrightarrow \exists a_1 \in A(\mathbf{A} \models \psi(a_1)[v]) \\ &\leftrightarrow \exists b_1 \in B(\mathbf{B} \models \psi(b_1)[\alpha(v)]) \\ &\leftrightarrow \mathbf{B} \models \varphi[\alpha(v)].\end{aligned}$$

$\square$