

**Theorem 1.** (*Ultrafilter Convergence Theorem*) Let  $\mathcal{X} = \langle X; \mathcal{T} \rangle$  be a topological space.

- (1)  $\mathcal{X}$  is compact iff for every ultrafilter  $\mathcal{U}$  on a set  $I$  and every function  $f: I \rightarrow X$  there is at least one  $\ell \in X$  such that  $\lim_{\mathcal{U}} f = \ell$ .
- (2)  $\mathcal{X}$  is Hausdorff iff for every ultrafilter  $\mathcal{U}$  on a set  $I$  and every function  $f: I \rightarrow X$  there is at most one  $\ell \in X$  such that  $\lim_{\mathcal{U}} f = \ell$ .

*Proof.* [(1),  $\Rightarrow$ ] Assume that  $\mathcal{X}$  is compact, but there is some ultrafilter  $\mathcal{U}$  on some set  $I$  and some function  $f: I \rightarrow X$  with no  $\ell$  satisfying  $\lim_{\mathcal{U}} f = \ell$ . Each  $x \in X$  must have an open neighborhood  $O_x$  such that  $f^{-1}(O_x) \notin \mathcal{U}$ . By compactness, the open cover  $\{O_x \mid x \in X\}$  has a finite subcover  $\{O_1, \dots, O_n\}$ . This yields  $\bigcup_{i=1}^n f^{-1}(O_i) = f^{-1}(\bigcup_{i=1}^n O_i) = I \in \mathcal{U}$ . We claim that if a finite union of subsets of  $I$  lies in  $\mathcal{U}$  then at least one summand lies in  $\mathcal{U}$ . Since no set  $f^{-1}(O_i)$  lies in  $\mathcal{U}$ , this claim produces a contradiction.

We prove the contrapositive of the claim just mentioned, namely that  $V, W \notin \mathcal{U}$  implies  $V \cup W \notin \mathcal{U}$ . If  $V, W \notin \mathcal{U}$ , then  $X \setminus V, X \setminus W \in \mathcal{U}$ , hence  $(X \setminus V) \cap (X \setminus W) = X \setminus (V \cup W) \in \mathcal{U}$ . Therefore  $V \cup W \notin \mathcal{U}$ .

[(1),  $\Leftarrow$ ] Choose a collection  $\mathcal{C}$  of closed subsets of  $\mathcal{X}$  with the FIP. Let  $I = X$  and let  $\mathcal{F}$  be the filter on  $I$  generated by  $\mathcal{C}$ . Let  $\mathcal{U}$  be an ultrafilter extending  $\mathcal{F}$ . Let  $f: I \rightarrow X$  be the identity function. Choose  $\ell$  so that  $\lim_{\mathcal{U}} f = \ell$ . Let  $\mathcal{N}_{\ell}$  be the set of open neighborhoods of  $\ell$ . If  $O \in \mathcal{N}_{\ell}$ , then  $O = f^{-1}(O) \in \mathcal{U}$ . This yields  $\mathcal{C} \cup \mathcal{N}_{\ell} \subseteq \mathcal{U}$ , showing that  $\mathcal{C} \cup \mathcal{N}_{\ell}$  has the FIP. This proves that  $\ell$  belongs to every closed set in  $\mathcal{C}$ , hence that  $\ell \in \bigcap \mathcal{C}$ .

[(2),  $\Rightarrow$ ] Assume  $\mathcal{X}$  is Hausdorff and that  $\lim_{\mathcal{U}} f = \ell$  and  $\lim_{\mathcal{U}} f = m$  for appropriate  $\mathcal{U}, f, \ell, m$  with  $\ell \neq m$ . Choose disjoint open neighborhoods  $O_{\ell}$  and  $O_m$  of  $\ell$  and  $m$  respectively. Then  $f^{-1}(O_{\ell})$  and  $f^{-1}(O_m)$  are disjoint sets in  $\mathcal{U}$ , a contradiction.

[(2),  $\Leftarrow$ ] Now suppose that  $\mathcal{X}$  is not Hausdorff and that  $\ell \neq m$  are elements of  $\mathcal{X}$  that cannot be separated by disjoint open sets. If  $\mathcal{N}_{\ell}$  and  $\mathcal{N}_m$  are the open neighborhood systems of these two points, then  $\mathcal{N}_{\ell} \cup \mathcal{N}_m$  has the FIP. Let  $I = X$ , let  $\mathcal{U}$  be an ultrafilter on  $I$  containing  $\mathcal{N}_{\ell} \cup \mathcal{N}_m$ , and let  $f: I \rightarrow X$  be the identity function. Both  $\lim_{\mathcal{U}} f = \ell$  and  $\lim_{\mathcal{U}} f = m$  hold, establishing the contrapositive of (2),  $\Leftarrow$ .  $\square$