Ultrafilters.

Definition 1. A ***filter*** on I is a nonempty set $\mathcal{F} \subseteq \mathcal{P}(I)$ such that

(1) $\mathcal{F}$ is closed under finite intersection, and
(2) $\mathcal{F}$ is closed under the formation of supersets (if $U \in \mathcal{F}$ and $U \subseteq V$, then $V \in \mathcal{F}$).

$\mathcal{F}$ is ***improper*** if $\mathcal{F} = \mathcal{P}(I)$ and is ***trivial*** if $\mathcal{F} = \{I\}$.

It follows from Definition 1 (2) that $\mathcal{F}$ is improper iff $\emptyset \in \mathcal{F}$.

Lemma 2. *The set of (proper) filters on I is closed under intersection and unions of directed families.*

Proof. The fact that filters are defined by closure properties guarantees that the set of (proper) filters on I is closed under intersection.

Let $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ be a Λ-directed family of filters on I. If $U_1, \ldots, U_k \in \bigcup \mathcal{F}_\lambda$, then these sets belong to $\mathcal{F}_\mu$ for some μ. But then

$$U_1 \cap \cdots \cap U_k \in \mathcal{F}_\mu \subseteq \bigcup \mathcal{F}_\lambda.$$

Now suppose that $U \in \bigcup \mathcal{F}_\lambda$ and $U \subseteq V$. Then $U \in \mathcal{F}_\mu$ for some μ and therefore $V \in \mathcal{F}_\mu \subseteq \bigcup \mathcal{F}_\lambda$. This shows that the union of a directed family of filters is a filter.

If $\bigcup \mathcal{F}_\lambda$ is improper, then it contains $\emptyset$. It must be that $\emptyset \in \mathcal{F}_\mu$ for some μ, and this $\mathcal{F}_\mu$ is also improper. □

For any set $\mathcal{S} \subseteq \mathcal{P}(I)$ there is a least filter containing $\mathcal{S}$, namely the intersection of all filters containing $\mathcal{S}$. This filter is called the ***filter generated by*** $\mathcal{S}$ and denoted $\langle \mathcal{S} \rangle$.

Lemma 3. *If $\mathcal{S} \subseteq \mathcal{P}(I)$, then TFAE.*

(1) *$U \in \langle \mathcal{S} \rangle$.*
(2) *U contains a finite intersection of elements of $\mathcal{S}$.*

Proof. [(1)⇒(2)] The set of all those sets which contain a finite intersection of elements of $\mathcal{S}$ is a filter containing $\mathcal{S}$.

[(2)⇒(1)] Any filter containing $\mathcal{S}$ must contain every set that contains a finite intersection of elements of $\mathcal{S}$ by Definition 1 (1) and (2). □

Definition 4. $\mathcal{S} \subseteq \mathcal{P}(I)$ has the ***finite intersection property (FIP)*** if any finite intersection of elements of $\mathcal{S}$ is nonempty.

Hence $\mathcal{S}$ has the finite intersection property iff $\langle \mathcal{S} \rangle$ is proper.

Definition 5. A proper filter $\mathcal{F}$ on I is an ***ultrafilter*** if for every $U \subseteq I$ either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$.

Lemma 6. (1) *A filter $\mathcal{F}$ on I is maximal under inclusion among proper filters on I iff it is an ultrafilter.*
(2) *(Ultrafilter Lemma) Every proper filter on I can be extended to an ultrafilter.*

(3) *(Strengthening of (2)) Every proper filter on I is the intersection of the ultrafilters that extend it.*

Proof. [Proof of (1)] Assume $\mathcal{F}$ is a maximal filter. If $U \notin \mathcal{F}$, then $\langle \mathcal{F} \cup \{U\} \rangle$ is improper, hence contains $\emptyset$. By Lemma 3, there is some $V \in \mathcal{F}$ such that $\emptyset = U \cap V$. Replacing V by a supserset if necessary we may assume that $V = I \setminus U$. Hence $U \notin \mathcal{F}$ implies that $I \setminus U \in \mathcal{F}$, showing that maximal filters are ultrafilters.

For the other direction in (1), observe that any proper extension of an ultrafilter must contain some set and its complement, hence must contain $\emptyset$. Thus ultrafilters are maximal.

[Proof of (2)] If $\mathcal{F}$ is a filter satisfying $\emptyset \notin \mathcal{F}$, then Zorn's Lemma guarantees that $\mathcal{F}$ can be extended to a filter $\mathcal{F}'$ that is maximal for $\emptyset \notin \mathcal{F}'$. (This uses the first part of Lemma 2.) By (1), any filter maximal for $\emptyset \notin \mathcal{F}'$ is an ultrafilter.

[Proof of (3)] Suppose that $\mathcal{F}$ is proper and $U \notin \mathcal{F}$. By repeating the argument from the first paragraph of part (1) we see that $\mathcal{F}' := \langle \mathcal{F} \cup \{I \setminus U\} \rangle$ is a proper filter. Extend $\mathcal{F}'$ to an ultrafilter $\mathcal{U}$ using part (2). Since $\mathcal{U}$ must contain $I \setminus U$ it cannot contain U. This shows that whenever $U \notin \mathcal{F}$ there is an ultrafilter $\mathcal{U}$ extending $\mathcal{F}$ satisfying $U \notin \mathcal{U}$. Hence the intersection of the ultrafilters containing $\mathcal{F}$ is a filter containing $\mathcal{F}$ which contains no sets not in $\mathcal{F}$, i.e. $\mathcal{F} = \bigcap_{\mathcal{U} \supseteq \mathcal{F}} \mathcal{U}$ $\square$