Preservation Theorems.

Theorem 1. *(Frayne, Morel, Scott) Let $\mathcal{K}\cup\{\mathbf{A}\}$ be a class of $\mathcal{L}$-structures. $\mathbf{A} \in Mod(Th(\mathcal{K}))$ iff $\mathbf{A}$ is elementarily equivalent to an ultraproduct of members of $\mathcal{K}$.*

Proof. [$\Rightarrow$] $\mathbf{A} \in \mathrm{Mod}(\mathrm{Th}(\mathcal{K}))$ holds if, in the space of $\mathcal{L}$-structures modulo equivalence, the class $\overline{\mathbf{A}}$ is a limit of a set $\{\overline{\mathbf{A}}_i \mid i \in I\}$ of classes represented by members of $\mathcal{K}$. That is, $\overline{\mathbf{A}} = \lim_{\mathcal{U}} \overline{\mathbf{A}}_i$ for some ultrafilter $\mathcal{U}$ and some structures $\mathbf{A}_i \in \mathcal{K}$. Choosing representatives of classes, this means that $\mathbf{A} \equiv \prod_{\mathcal{U}} \mathbf{A}_i$.

[$\Leftarrow$] Here we need to show that $\mathrm{Mod}(\mathrm{Th}(\mathcal{K}))$ is closed under elementary equivalence and ultraproducts. For the part about elementary equivalence, $\mathbf{A} \equiv \mathbf{B} \in \mathrm{Mod}(\mathrm{Th}(\mathcal{K}))$ holds iff $\mathrm{Th}(\mathbf{A}) = \mathrm{Th}(\mathbf{B}) \supseteq \mathrm{Th}(\mathcal{K})$, which implies $\mathbf{A} \in \mathrm{Mod}(\mathrm{Th}(\mathcal{K}))$. The part about ultraproducts follows from Łos's Theorem. $\square$

Corollary 2. *An axiomatizable class $\mathcal{K}$ fails to be finitely axiomatizable iff it contains an ultraproduct of structures from the complement $\mathcal{K}^c$.*

Proof. $\mathcal{K}$ is axiomatizable iff it is closed under elementary equivalence and ultraproducts. If this holds, the complementary class $\mathcal{K}^c$ must be closed under elementary equivalence. Therefore, $\mathcal{K}^c$ will be axiomatizable iff it is closed under ultraproducts. The corollary follows from this. $\square$

Theorem 3. *(F-M-S) Let $\mathcal{K} \cup \{\mathbf{A}\}$ be a class of $\mathcal{L}$-structures. $\mathbf{A} \in Mod(Th(\mathcal{K}))$ iff $\mathbf{A}$ is elementarily embeddable in an ultraproduct of members of $\mathcal{K}$.*

Proof. Given Theorem 1, the only thing left to prove is that if $\mathbf{A}$ is elementarily equivalent to an an ultraproduct of structures in $\mathcal{K}$, then $\mathbf{A}$ is elementarily embeddable in an ultraproduct of structures in $\mathcal{K}$.

Suppose that $\mathbf{A} \equiv \mathbf{B} = \prod_{\mathcal{U}} \mathbf{A}_i$. We shall elementarily embed $\mathbf{A}$ into an ultrapower of $\mathbf{B}$. Since ultraproducts of ultraproducts are ultraproducts (HW problem), this will complete the proof.

Our goal is to prove that if $\mathbf{A} \equiv \mathbf{B}$, then $\mathbf{A}$ is elementarily embeddable in an ultrapower of $\mathbf{B}$.

Let I be the set of finite partial injections from A to B. For each tuple $\mathbf{a} \in A^n$ and each formula $\varphi = \varphi(x_1, \ldots, x_n)$ such that $\mathbf{A} \models \varphi[\mathbf{a}]$ let $U_{(\mathbf{a},\varphi)}$ be the subset of I consisting of all $i \in I$ such that $\mathbf{a} \subseteq \mathrm{dom}(i)$ and $\mathbf{B} \models \varphi[i(\mathbf{a})]$. Let $\mathcal{S} \subseteq \mathcal{P}(I)$ be the collection of all such sets.

Claim 4. *$\mathcal{S}$ has the finite intersection property.*

Since $U_{(\mathbf{a},\varphi(\mathbf{x}))} \cap U_{(\mathbf{b},\psi(\mathbf{y}))} = U_{(\mathbf{ab},\varphi(\mathbf{x})\wedge\psi(\mathbf{y}))}$, it suffices to show that no set $U_{(\mathbf{a},\varphi)}$ is empty. For this, choose any $U_{(\mathbf{a},\varphi)} \in \mathcal{S}$. $\mathbf{A} \models \varphi[\mathbf{a}]$ implies $\mathbf{A} \models \exists\mathbf{x}\varphi(\mathbf{x}) \wedge \delta(\mathbf{x})$ where $\delta(\mathbf{x})$ is the conjunction of $(x_i \doteq x_j)$ if $a_i = a_j$ and $(\neg(x_i \doteq x_j))$ if $a_i \neq a_j$. Since $\mathbf{A} \equiv \mathbf{B}$, $\mathbf{B} \models \exists\mathbf{x}\varphi(\mathbf{x}) \wedge \delta(\mathbf{x})$, so there is a tuple $\mathbf{b} \in B^n$ with the same pattern of equalities as $\mathbf{a}$ such that $\mathbf{B} \models \varphi[\mathbf{b}]$. There is a finite partial injection i such that $i(\mathbf{a}) = \mathbf{b}$ showing that $i \in U_{(\mathbf{a},\varphi)}$.

Let $\mathcal{U}$ be an ultrafilter extending $\mathcal{S}$. Extend each partial injection $i \in I$ to a function $\hat{i}\colon A \to B$ in an arbitrary way. This yields a product function $\prod_{i\in I} \hat{i}\colon A \to B^I$ which, upon composing with the natural map modulo the ultrafilter congruence, induces a function $\pi\colon A \to \prod_{\mathcal{U}} B$.

Claim 5. $\pi\colon \mathbf{A} \to \prod_{\mathcal{U}} \mathbf{B}$ *is an elementary embedding.*

If, say, $\mathbf{A} \models \varphi[\mathbf{a}]$, then for each $i \in U_{(\mathbf{a},\varphi)}$ it is the case that $\mathbf{B} \models \varphi[i(\mathbf{a})] = \varphi[\hat{i}(\mathbf{a})]$. By Łos's Theorem, $\prod_{\mathcal{U}} \mathbf{B} \models \varphi[\pi(\mathbf{a})]$, so π is an elementary embedding. □

Corollary 6. *(F-M-S) The following are equivalent for a class $\mathcal{K}$ of $\mathcal{L}$-structures.*

(1) *$\mathcal{K}$ is elementary.*
(2) *$\mathcal{K}$ is closed under ultraproducts and elementary equivalence.*
(3) *$\mathcal{K}$ is closed under ultraproducts and the formation of elementary substructures.*

Proof. This follows from Theorems 1 and 3. □

Theorem 7. *(Los-Tarski Preservation Theorem) A class of $\mathcal{L}$-structures is axiomatizable by universal sentences iff it is closed under the formation of substructures and ultraproducts.*

Theorem 8. *An axiomatizable class of $\mathcal{L}$-structures has a set of existential axioms iff it is closed under the formation of extensions.*

Theorem 9. *(Chang-Los-Suszko) An axiomatizable class of $\mathcal{L}$-structures has a set of $\forall\exists$ axioms iff it is closed under unions of chains.*

Theorem 10. *(Birkhoff) If $\mathcal{L}$ is an algebraic language, then a class of $\mathcal{L}$-structures is axiomatizable by universally quantified atomic sentences iff it is closed under the formation of homomorphic images, subalgebras and products.*