Linear Algebra
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. For each of the following statements, answer True or False. Give a brief (1-sentence) justification for each answer.

(a) Every homogeneous system is consistent.

True. $\mathbf{x} = \mathbf{0}$ is a solution to $A\mathbf{x} = \mathbf{0}$.

(b) The product of two elementary matrices is an elementary matrix.

False. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are elementary, but their product is $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, which is not elementary.

(c) A subset of an independent set of vectors is independent.

True. To justify this, we explain why the contrapositive statement is true, namely: if $X \subseteq Y$ and X is <u>dep</u>endent, then Y is <u>dep</u>endent.

Assume $X \subseteq Y$. If X is dependent, then there is a nontrivial dependence relation among the vectors in X. This is also a nontrivial dependence relation among the vectors in Y, so Y is dependent.

2. Use linear algebra to find a curve of the form $y = ax^2 + bx + c$ that passes through the points $(x, y) = (-1, -2), (0, 1)$ and $(1, 8)$. (Hint: start by plugging the points into the equation.)

Plugging points leads to the system

$$\begin{array}{cccc} a & -b & +c & = -2 \\ 0 & +0 & +c & = 1 \\ a & +b & +c & = 8 \end{array},$$

which has the unique solution $a = 2, b = 5, c = 1$. Hence the curve is the graph of $y = 2x^2 + 5x + 1$.

3. (i) Find a basis and (ii) determine the dimension of the subspace $V \leq \mathbb{R}^4$ consisting of vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ whose cooordinates sum to zero. ($V = \{\mathbf{x} \mid x_1 + x_2 + x_3 + x_4 = 0\}$)

V is the set of solutions to the system $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$. The matrix involved is already in reduced row echelon form. The free variables are x_2, x_3, x_4 while x_1 is basic. The general solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for V is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. This shows that $\dim(V) = 3$.

4. Explain why if A is invertible, then A^T is invertible.

Transposing the equations $AA^{-1} = I$ and $A^{-1}A = I$ yields $(A^{-1})^T A^T = I^T = I = A^T (A^{-1})^T$, so $(A^{-1})^T$ is the inverse of A^T.