

LINEAR ALGEBRA (MATH 3130): REVIEW SHEET

Sections 1.1–4.7, excluding 1.4, 1.6, 2.6, 3.8, 3.10, 4.6

I. Systems of linear equations.

- (a) Augmented matrix and coefficient matrix of a system.
- (b) Row reduction. (Reduced) row echelon form. Pivots, pivot positions, pivot columns.
- (c) Free and basic(=pivot) variables. Solutions sets. Parametrized form of a solution.
- (d) Consistent and inconsistent systems.
- (e) Homogeneous systems. Relationship between solutions of $A\mathbf{x} = \mathbf{b}$ and solutions of $A\mathbf{x} = \mathbf{0}$.

II. Matrix arithmetic.

- (a) Matrices can be added, negated, multiplied with each other, and scaled, provided the dimensions are right.
- (b) Laws of matrix arithmetic resemble the laws of \mathbb{R} , except matrix multiplication may be noncommutative.
- (c) Matrix transpose.
- (d) Left and right inverses. (Equivalent properties.) Two-sided inverses. A 1-sided invertible matrix is 2-sided invertible iff it is square. Nonsingularity.
- (e) Elementary matrices. Row reduction is expressible as left multiplication by a sequence of elementary matrices. Matrices A and B are row equivalent iff $A = LB$ for some invertible matrix L .
- (f) Algorithm for finding inverses.

III. Vectors and vector spaces.

- (a) Linear systems may be viewed as vector equations.
- (b) Definition of vector space. Definition of subspace.
- (c) Geometric interpretation of vector space operations.
- (d) Definition of column space and nullspace of a matrix.
- (e) Spanning set of vectors.
- (f) Linearly (in)dependent set of vectors.
- (g) Bases and dimension.
- (h) Rank of a matrix.
- (i) The four fundamental subspaces.
- (j) Algorithms for finding bases for the column space and nullspace.
- (k) Rank plus nullity theorem.
- (l) Algorithms for extending a basis for a subspace to a basis for a larger subspace, for finding the basis for a sum of two subspaces and for finding a basis for an intersection of two subspaces.
- (m) $\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$.

IV. Linear Transformations.

- (a) Definition. Fact that any linear transformation has the form $T(\mathbf{x}) = A\mathbf{x}$.
- (b) The problem of solving the linear system $A\mathbf{x} = \mathbf{b}$ may be viewed as the problem of finding a vector $\mathbf{x} \in T^{-1}(\mathbf{b})$ for $T(\mathbf{x}) = A\mathbf{x}$.

General advice on preparing for a math test.

Be prepared to demonstrate understanding in the following ways.

- (i) Know the definitions of new concepts, and the meanings of the definitions.
- (ii) Know the statements and meanings of the major theorems.
- (iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
- (iv) Know how to perform the different kinds of calculations discussed in class.
- (v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
- (vi) Know how to correct mistakes made on old HW.

Sample Problems.

- (1) Find the inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ in two different ways: first by using the formula for inverses of 2×2 matrices, and then by using Gaussian elimination.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 4 - 3 \cdot 2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

- (2) Find the general solution to $A\mathbf{x} = \mathbf{b}$ if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Which variables are free and which are basic?

x_2 is free. $x_3 = -1$ and $x_1 = 4 - 2x_2$.

- (3) Let A be a square matrix. Explain why if the columns of A are independent, then the columns of A^2 are independent.

Solution 1: If the columns of A are independent, then $N(A) = \{\mathbf{0}\}$. If $\mathbf{a} \in N(A^2)$, then $A^2\mathbf{a} = \mathbf{0}$, so $A\mathbf{a} \in N(A) = \{\mathbf{0}\}$, so $\mathbf{a} = \mathbf{0}$. Hence $N(A^2) = \{\mathbf{0}\}$, so the columns of A are independent.

Solution 2: If the columns of A are independent, then A is invertible. Then A^2 is also invertible (with inverse $(A^{-1})^2$), so the columns of A^2 are independent.

- (4) Show that if $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is independent, then $\{\mathbf{a}, \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b} + \mathbf{c}\}$ is also independent.

Consider a possible dependence relation among the vectors in $X = \{\mathbf{a}, \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b} + \mathbf{c}\}$:

$$\mathbf{0} = r_1 \cdot \mathbf{a} + r_2 \cdot (\mathbf{a} + \mathbf{b}) + r_3 \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = (r_1 + r_2 + r_3) \cdot \mathbf{a} + (r_2 + r_3) \cdot \mathbf{b} + r_3 \cdot \mathbf{c}.$$

$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is independent, so $r_1 + r_2 + r_3 = r_2 + r_3 = r_3 = 0$, and therefore $r_1 = r_2 = r_3 = 0$. Since the only dependence relation among the vectors in X is trivial, X is independent.

- (5) Explain why if A and B are $n \times n$ matrices satisfying $A\mathbf{x} = B\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.

If $A\mathbf{x} = B\mathbf{x}$ for all vectors \mathbf{x} , then $A\mathbf{e}_i = B\mathbf{e}_i$ for all i . But $A\mathbf{e}_i$ is the i th column of A and $B\mathbf{e}_i$ is the i th column of B , so A and B have the same columns. This forces $A = B$.

- (6) Let A and B be $m \times n$ matrices. Explain why $A \stackrel{\text{row}}{\sim} B$ iff A and B have the same nullspace. (Hint: See the Feb 3 lecture notes.)

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- (7) Among all $n \times n$ matrices whose entries are all either 0 or 1 what is the maximum possible number of 1's if the matrix is invertible? What is the minimum number of 1's if the matrix is invertible? For which values of n is it possible for the number of 1's to be equal to the number of 0's and still have the matrix invertible?

An invertible matrix must have linearly independent columns and rows, so all columns and rows must be different. Thus, we cannot have two columns (or rows) whose entries are all 1's. This means that there must be at least $n - 1$ zeros in an invertible $n \times n$ 0, 1-matrix.

But we don't need more than $n - 1$ zeros, since the matrix $\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$ is invertible

and has only $n - 1$ zeros. (Check invertibility by applying Gaussian Elimination.)

Similarly, an invertible 0, 1-matrix must have at least one 1 in each row and column, so at least n 1's. But we don't need more than n 1's, since I_n is invertible and has only n 1's.

- (8) Explain why if $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a subset of \mathbb{R}^n that is linearly independent and spans the space, then $k = n$.

The row reduced form of $[S] = [\mathbf{v}_1 \cdots \mathbf{v}_k]$ must have a pivot in every row and column, so the number of vectors in S must equal the number of entries in the vector, i.e. n .

- (9) Which matrices commute with $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$? (Hint: set up a linear system.)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ iff } \begin{array}{rcl} a + 3b & = & a + 2c \\ 2a + 4b & = & b + 2d \\ c + 3d & = & 3a + 4c \\ 2c + 4d & = & 3b + 4d \end{array}$$

You must solve this system. (d and c are free, $b = \frac{2}{3}c$, $a = d - c$.)

- (10) Explain why the set of columns of an $n \times n$ invertible matrix spans \mathbb{R}^n . Then explain why this set of columns is independent.

If A is invertible, then for any $\mathbf{b} \in \mathbb{R}^n$ the vector $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution to $A\mathbf{x} = \mathbf{b}$. This expression, written as a vector equation, shows how to express \mathbf{b} as a linear combination of the columns of A . Since \mathbf{b} was chosen arbitrarily, any vector in \mathbb{R}^n is a linear combination of the columns of A .