LINEAR ALGEBRA (MATH 3130): REVIEW SHEET 2

From the book: Sections 4.6–5.6 (excluding 4.9 and 5.2) together with Sections 5.11, 5.13 and 6.1–7.2.

IV. More on Linear Transformations.
- (c) One-to-one and onto linear transformations. Isomorphisms.
- (d) Finding coordinates relative to a basis.
- (e) Every finitely generated real vector space is isomorphic to $\mathbb{R}^n$ for some n.
- (f) Finding the standard matrix of a transformation.
- (g) Finding the standard matrix for plane rotations and reflections.
- (h) Matrices under change of basis.

V. Applications.
- (a) Network flow.
- (b) Balancing chemical equations.
- (c) Modeling the movement of goods in a simple economy. Stochastic matrices.

VI. General fields and vector spaces.
- (a) Definition of "field". Examples $\mathbb{Q}$, $\mathbb{Q}[\sqrt{2}]$, $\mathbb{R}$, $\mathbb{C} = \mathbb{R}[i]$ and $\mathbb{F}_2$.
- (b) Arithmetic of $\mathbb{C}$, including $\overline{\alpha}$, $|\alpha|$ and $\arg(\alpha)$.
- (c) Geometrical interpretation of the arithmetic operations of $\mathbb{C}$.

VII. Inner product spaces, orthogonality, least squares.
- (a) Inner products in general and dot product in particular. Norms. (Arithmetic of dot product follows from that of matrices, since $\mathbf{u} \bullet \mathbf{v} = \mathbf{u}^T\mathbf{v}$.)
- (b) Length, distance and angle in $\mathbb{R}^n$ and $\mathbb{C}^n$.
- (c) How to find unit vectors in a given direction in $\mathbb{R}^n$ and $\mathbb{C}^n$.
- (d) Cauchy-Bunyakovsky-Schwarz inequality. Fact that any inner product on V induces a norm on V.
- (e) There is no positive definite, symmetric, bilinear form definable on $\mathbb{C}$. Antilinear functions, sesquilinear forms and complex inner products.
- (f) Orthogonality. Orthogonal complement of a subspace.
- (g) Orthogonal projection onto a subspace.
- (h) Gram-Schmidt orthonormalization.
- (i) Orthogonal and unitary matrices. Rotations in 3-space.
- (j) Approximate solutions to $A\mathbf{x} = \mathbf{b}$ via least squares. Normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$. Fitting curves to data.

VIII. The determinant.
- (a) Signed volume.
- (b) Minor, cofactor, definition of the determinant via the Laplace expansion.
- (c) $\det(A)$ is defined only if A is square. $\det(A) \neq 0$ iff the columns of A are independent.
- (d) Adjugate matrix. Fact that $A \cdot \mathrm{adj}(A) = \det(A) \cdot I$, hence $A^{-1} = (1/\det(A))\mathrm{adj}(A)$ when A is invertible.
- (e) Further properties: $\det(AB) = \det(A)\det(B)$, the determinant can be computed by Gaussian elimination, the determinant of a block triangular matrix is the product of the determinants of the blocks, if $T(\mathbf{x}) = A\mathbf{x}$, then the determinant of A measures the "volume expansion" associated with T.
- (f) "Correct" definition: the determinant is the unique alternating multilinear function d of n variables defined on $\mathbb{R}^n$ for which $d(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

IX. Eigenvalues, eigenvectors, eigenspaces.
- (a) Eigenvectors identify "preserved directions" of a linear transformation $T\colon V \to V$.
- (b) Definitions of eigenvector, eigenvalue, eigenspaces.

(c) Methods of calculation: characteristic polynomial $\chi_A(\lambda)$ equals $\det(\lambda I - A)$; e-values of A are the roots of $\chi_A(\lambda) = 0$; e-space V_λ equals $\mathrm{Null}(\lambda I - A)$; λ-eigenvectors are the nonzero vectors of V_λ. Fast calculation of e-values for (block) triangular matrices.
(d) Proof that every rotation of 3-space has an axis.

X. Diagonalization.
(a) Structure of roots of a real polynomial over $\mathbb{R}$ or $\mathbb{C}$, and of a complex polynomial over $\mathbb{C}$. Algebraic multiplicity of an e-value.
(b) Geometric multiplicity of an e-value.
(c) Defn. of "diagonalizable". Thm. A transformation $T\colon V \to V$ is diagonalizable iff V has a basis consisting of e-vectors for T iff the geometric multiplicity of each e-value equals its algebraic multiplicity.
(d) Similarity: A is similar to B if A is a conjugate of B, i.e., $A = S^{-1}BS$. Similarity is an equivalence relation on the set of $n \times n$ matrices. Matrices are similar iff they represent the same transformation relative to different bases. Similar matrices have the same characteristic polynomial, hence same e-values. If $A = S^{-1}BS$, then $S\colon V_\lambda^A \to V_\lambda^B$ is an isomorphism for each e-value λ. A is diagonalizable iff it is similar to a diagonal matrix.
(e) Raising matrices to powers via diagonalization. Solving linear recurrences.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) Computational problems:
(a) Find bases for the null space, row space and column space of the 3×3 matrix whose entries are all 1. What are the dimensions of these spaces?
(b) Put the numbers $1, 2, \ldots, 9$ into a 3×3 matrix in order. What is the determinant?
(c) Find a change of basis matrix from $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$ and $\mathcal{C} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.
(d) Find the characteristic equation, e-values, and e-spaces of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find a matrix S that conjugates A into diagonal form.
(e) Find a basis for $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}^{\perp}$.

(2) Can any of the following exist? (If so, give an example, if not give a reason.)

(a) A vector space with an empty basis.
(b) A matrix of rank zero.
(c) A matrix with no determinant.
(d) A matrix with a zero dimensional eigenspace.
(e) An invertible matrix whose row sums are all zero.
(f) A real matrix whose null space equals its column space.
(g) A matrix A such that $\text{nullity}(A) = 1$ and $\text{nullity}(A^2) = 3$.
(h) A matrix where the dimension of the row space is greater than the dimension of the column space.
(i) A real number that does not arise as the determinant of a real matrix.
(j) A vector space with no subspaces.
(k) An isomorphism between vector spaces of different dimensions.
(l) A matrix whose row space is isomorphic to its column space.
(m) A matrix whose characteristic polynomial is $\lambda^2 + \lambda + 1$.
(n) An eigenvalue whose geometric multiplicity exceeds its algebraic multiplicity.
(o) A real 10×10 matrix with only one eigenvector.
(p) A matrix equal to its adjugate.
(q) A left stochastic matrix that is not right stochastic.
(r) Conjugate matrices of different ranks.
(s) Conjugate matrices that are not similar.
(t) A matrix that is diagonalizable over $\mathbb{C}$ but not diagonalizable over $\mathbb{R}$.
(u) An orthogonal basis for $\mathbb{R}^3$ that is not orthonormal.
(v) A nondiagonalizable complex matrix.
(w) An orthogonal matrix with determinant zero.
(x) A 3×3 orthogonal matrix with no zero entries.
(y) Subspaces U and W such that $U + W \neq U \oplus W$.
(z) A real vector that is orthogonal to itself.

(3) Give the dimensions of the following real vector spaces.
(a) The space of real polynomials $p(t)$ of degree at most 3 which satisfy $p(1) = p(-1) = 0$.
(b) The space of 3×3 upper triangular real matrices.
(c) The space of twice continuously differentiable functions $y = f(x)$ satisfying $y'' = 0$.

(4) How would you solve the following problem? Suppose that V has basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and U is a subspace of V with basis $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$. How do you find a basis for V whose first m vectors form a basis for U?

(5) Suppose that you are given bases $\mathcal{B}$ and $\mathcal{C}$ for subspaces U and W of a space V. How would you find a basis for $U + W$? How would you find a basis for $U \cap W$? (Hint: in both cases, you should apply Gaussian Elimination to the matrix $[\mathcal{B}|\mathcal{C}]$. How should you use the results?)

(6) Is there a 3×3 matrix whose minors are nonzero and all equal? Is there a 3×3 matrix whose cofactors are nonzero and all equal?

(7) Let S be a 2×2 invertible matrix. Consider the linear transformation of "conjugation by S":

$$T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})\colon A \mapsto S^{-1}AS.$$

Show that if λ is an e-value of T, then so is λ^k for any k. Show that 0 is not an e-value of T. Explain why the e-values of T can only be $+1$ or -1. Show that $+1$ occurs as an e-value with multiplicity at least 2.

(8) What is the characteristic polynomial for the $n \times n$ matrix whose entries are all 1?

(9) Show that $\dim(U+W) = \dim(U)+\dim(W)-\dim(U\cap W)$. (Solution 1 hint: choose a basis for $U\cap W$ and extend it in different ways to bases for both U and W. Show that all the vectors together form a basis for $U+W$.) (Solution 2 hint: let $\mathcal{B}$ and $\mathcal{C}$ be bases for U and W. Apply the rank+nullity theorem to the matrix $[\mathcal{B}|\mathcal{C}]$.)

(10) The points $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are the vertices of a regular tetrahedron. Find the lengths of the sides and the angles formed by adjacent faces.

(11) Find the least squares curve of the form $y = ax^2 + bx + c$ that best fits the data points $(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)$.

(12) Show that if V is finite dimensional and U is a subspace, then $V = U \oplus U^\perp$.

(13) Show that $(U + W)^\perp = U^\perp \cap W^\perp$.

(14) Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$.