Practice Problems. linear combinations, subspaces, span, linear (in)dependence, basis, dimension.

(1) Which is right: "dependent" or "dependant"?

(2) Let A be an $m \times n$ matrix. Show that the set of vectors $\mathbf{b}$ which make $A\mathbf{x} = \mathbf{b}$ a consistent system is a subspace of $\mathbb{R}^m$.

(3) Show that a 1-element set $\{\mathbf{v}\}$ is linearly independent iff $\mathbf{v} \neq \mathbf{0}$. Show that a 2-element set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent iff neither is a scalar multiple of the other.

(4) Show that $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

(5) Show that $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is linearly dependent.

(6) Explain why a set of columns $\{[\mathbf{a}_1], \ldots, [\mathbf{a}_n]\} \subseteq \mathbb{R}^m$ is linearly independent iff the $m \times n$ matrix $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ has a left inverse. (It might be better to think about this in terms of solutions to $A\mathbf{x} = \mathbf{0}$.)

(7) Explain why a set of columns $\{[\mathbf{a}_1], \ldots, [\mathbf{a}_n]\} \subseteq \mathbb{R}^m$ spans $\mathbb{R}^m$ iff the $m \times n$ matrix $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ has a right inverse. (It might be better to think about this in terms of solutions to $A\mathbf{x} = \mathbf{b}$.)

(8) Use the last two problems to develop a test for when a subset of $\mathbb{R}^m$ is a basis.

(9) Use problems (6) and (7) to explain how to enlarge a linearly independent subset of $\mathbb{R}^m$ to a basis, and also how to shrink a spanning subset of $\mathbb{R}^m$ to a basis.

(10) Explain why a subset of an independent set is an independent set.

(11) Explain why a superset of a spanning set is a spanning set.

(12) Suppose that X and Y are linearly independent. Under what circumstances will $X \cup Y$ be linearly independent?