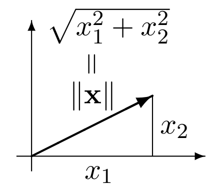
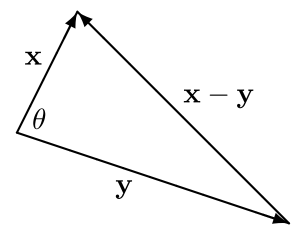
Length, distance and angle



(1) The computation of length is based on the Pythagorean Theorem: In $\mathbb{R}^n$ we use the formula

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

(2) The distance between (the endpoints of) $\mathbf{x}$ and $\mathbf{y}$ is computed $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}$.



(3) The computation of angle is based on the Law of Cosines:

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\|\mathbf{x}\|\|\mathbf{y}\| \cos(\theta).$$

(Compute the lengths and then solve for θ.)

Dot product

The function $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ (or even its square $\|\mathbf{x}\|^2 = x_1^2 + \cdots + x_n^2$) is nonlinear, and so hard to compute with symbolically. It is easier to compute symbolically with the dot product, defined

$$\mathbf{x} \bullet \mathbf{y} = \mathbf{x}^T \mathbf{y} = x_1 y_1 + \cdots + x_n y_n.$$

The dot product is linear in each variable separately (= bilinear), and length can be computed from the dot product using the formula $\|\mathbf{x}\|^2 = \mathbf{x} \bullet \mathbf{x}$.

For example, $\|\mathbf{x} - \mathbf{y}\|^2 = (\mathbf{x} - \mathbf{y}) \bullet (\mathbf{x} - \mathbf{y}) = \mathbf{x} \bullet \mathbf{x} - 2(\mathbf{x} \bullet \mathbf{y}) + \mathbf{y} \bullet \mathbf{y} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2(\mathbf{x} \bullet \mathbf{y})$. Comparing this to the Law of Cosines, we get a nicer formula for angle:

$$\cos(\theta) = \frac{\mathbf{x} \bullet \mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}.$$

Norms and inner products

The function $\|\mathbf{x}\| = (\mathbf{x} \bullet \mathbf{x})^{\frac{1}{2}}$ is an example of a norm on $\mathbb{R}^n$. Here is the general definition: A ***norm*** on a real vector space V is a function $\mathbf{n}\colon V \to \mathbb{R}$ that is

(1) (positive definite) $\mathbf{n}(\mathbf{v}) \geq 0$, with equality iff $\mathbf{v} = \mathbf{0}$,
(2) (homogeneous) $\mathbf{n}(r \cdot \mathbf{v}) = |r| \cdot \mathbf{n}(\mathbf{v})$, and
(3) (triangle inequality) satisfies $\mathbf{n}(\mathbf{u} + \mathbf{v}) \leq \mathbf{n}(\mathbf{u}) + \mathbf{n}(\mathbf{v})$.

The function $\|\mathbf{x}\| = (\mathbf{x}\bullet\mathbf{x})^{\frac{1}{2}} = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ is called the ***Euclidean norm*** in $\mathbb{R}^n$. Another example of a norm on $\mathbb{R}^n$ is the p-norm: $\|\mathbf{x}\|_p = (\sum_{i=1}^{n} x_i^p)^{\frac{1}{p}}$ for $p \geq 1$. By letting p go to infinity we obtain the ∞-norm: $\|\mathbf{x}\|_\infty = \max(|x_1|, \ldots, |x_n|)$.

The dot product is an example of an inner product. Here is the general definition: A ***real inner product*** on a real vector space V is a function $\langle -, - \rangle \colon V \times V \to \mathbb{R}$ that is

(1) (positive definite) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ with equality iff $\mathbf{v} = \mathbf{0}$,
(2) (symmetric) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$, and
(3) (bilinear) both $\langle \mathbf{u}, - \rangle$ and $\langle -, \mathbf{v} \rangle$ are linear transformations from V to $\mathbb{R}$.

Every inner product yields a norm via the definition $\|\mathbf{x}\| := \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$, but not every norm comes from an inner product. For example, the p-norm comes from an inner product iff $p = 2$.