Bases for Subspaces

We have discussed algorithms for finding bases for the column space and nullspace of a matrix. Here we describe how to use those algorithms to find bases for other spaces.

Extending a basis of U to a basis for V when $U \leq V$.
Here we assume that U is a subspace of V, $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a basis for U and $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V. Our goal is to find a basis for V whose first m vectors form a basis for U.

Algorithm: apply the column space algorithm to $A := [\mathcal{B}|\mathcal{C}] = [\mathbf{u}_1 \cdots \mathbf{u}_m|\mathbf{v}_1 \cdots \mathbf{v}_n]$.

Why it works: The column space of A is V. The columns of $\mathcal{B}$ are independent, so the first m columns of the reduced form of $[\mathcal{B}|\mathcal{C}]$ will be pivot columns. Hence the first m columns of A will be the first m columns of the basis obtained. (Note: the assumption that $\mathcal{C}$ is a basis for V can be weakened to the assumption that $\mathcal{B} \cup \mathcal{C}$ is a spanning set for V.)

Finding a basis for $U + W$.
If U and W are subspaces, then their ***sum*** is defined to be $U+W = \{\mathbf{u}+\mathbf{w} \mid \mathbf{u} \in U, \mathbf{w} \in W\} = \operatorname{span}(U \cup W)$. Assume that $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a basis for U and $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a basis for W. Our goal is to find a basis for $U + W$.

Algorithm: apply the column space algorithm to $A := [\mathcal{B}|\mathcal{C}]$.

Why it works: If you define $V = U+W$, then $U \leq V$. Now this algorithm is just a special case of the preceding one.

Finding a basis for $U \cap W$. The intersection of two subspaces is a subspace. Assume that $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ is a basis for U and $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a basis for W. Our goal is to find a basis for $U \cap W$.

Algorithm: apply the nullspace algorithm to $A := [\mathcal{B}|\mathcal{C}]$. If $N(A)$ has basis

$$\left\{ \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{y}_1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{x}_s \\ \mathbf{y}_s \end{bmatrix} \right\},$$

then $\{[\mathcal{B}]\mathbf{x}_1, \ldots, [\mathcal{B}]\mathbf{x}_s\}$ is a basis for $V \cap W$.

Why it works: This will be shown later.

Theorem 1. *If U and W are subspaces, then*

$$\dim(U+W) + \dim(U \cap W) = \dim(U) + \dim(W).$$

Proof. Apply the rank+nullity theorem to $[\mathcal{B}|\mathcal{C}]$, where $\mathcal{B}$ is a basis for U and $\mathcal{C}$ is a basis for W. $\square$

(1) Extend $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}$ to a basis for $\mathbb{R}^4$.

(2) Find a basis for the intersection of the subspaces spanned by $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$.

(3) Use Theorem 1 to prove that if U and W are 2-dimensional subspaces of $\mathbb{R}^3$ then there is a nonzero vector contained in both U and W.