Linear Algebra
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. For each of the following statements, answer True or False. Give a brief (1-sentence) justification for each answer.

(a) Every homogeneous system is consistent.

True. Any homogeneous system $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as a solution.

(b) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent if the vector $\mathbf{u}$ is not a linear combination of the others in the set.

False. If $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is not linearly independent even though $\mathbf{u}$ is not a linear combination of the others.

(c) The zero function $T(\mathbf{x}) = \mathbf{0}$ is a linear transformation.

True. The zero transformation satisfies $T(\mathbf{x}+\mathbf{y}) = \mathbf{0} = \mathbf{0}+\mathbf{0} = T(\mathbf{x})+T(\mathbf{y})$ and $T(r\mathbf{x}) = \mathbf{0} = r\mathbf{0} = rT(\mathbf{x})$.

(d) Upper triangular matrices are invertible.

False. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is upper triangular, but not invertible.

2. Use linear algebra to find a curve of the form $y = ax^2 + bx + c$ that passes through the points $(x, y) = (-1, 8), (0, 1)$ and $(1, -2)$. (Hint: start by plugging the points into the equation.)

Substituting points yields a system of linear equations in the unknowns a, b, c that has the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -2 \end{array}\right].$$

The solution is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$, which yields $y = 2x^2 - 5x + 1$.

3. Explain why if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a subset of $\mathbb{R}^n$ that is linearly independent and spans the space, then $k = n$.

Let A be the matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_k$. A has a left inverse, since S is linearly independent. A has a right inverse, since S spans $\mathbb{R}^n$. Hence A has a 2-sided inverse, hence A is square, hence $k = n$.

4. Explain why if A is invertible, then $A^T A$ is invertible.

If A is invertible (say with inverse B), then A^T is also invertible (with inverse B^T). The product $A^T A$ of two invertible matrices is again invertible (with inverse BB^T).