

## LINEAR ALGEBRA (MATH 3130): REVIEW SHEET

From the book: Sections 2.7–6.5, excluding Sections 4.8, 5.6–5.8.

### VII. Affine transformations.

- (a) An affine transformation has the form  $T(\mathbf{x}) = A\mathbf{x} + \mathbf{t}$ . Examples: translations, plane reflections about an arbitrary line, and plane rotations about an arbitrary point.
- (b) Affine transformations in  $\mathbb{R}^n$  may be represented in homogeneous coordinates in  $\mathbb{R}^{n+1}$  by matrices of the form  $\left[ \begin{array}{c|c} A & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array} \right]$ .

### VIII. Subspaces

- (a) The structure of subspaces of  $\mathbb{R}^n$ .
- (b) Four fundamental subspaces:  $\text{Col}(A)$ ,  $\text{Nul}(A)$ ,  $\text{Row}(A) = \text{Col}(A^T)$ , and  $\text{Nul}(A^T)$ .
- (c) Ordered and unordered bases for a subspace.
- (d) We gave algorithms for finding bases for  $\text{Nul}(A)$ ,  $\text{Row}(A)$ , and  $\text{Col}(A)$ .
- (e) Standard basis for  $\mathbb{R}^n$ .
- (f) Dimension of a subspace.
- (g) Rank and nullity of a matrix. Rank + nullity theorem.
- (h) Proof that “dimension” is well defined, namely, that the size of any independent set is less or equal the size of any spanning set, and that a maximal independent set is spanning while a minimal spanning set is independent.

### IX. The determinant.

- (a) Signed volume.
- (b) Minor, cofactor, definition of the determinant via the Laplace expansion.
- (c)  $\det(A)$  is defined only if  $A$  is square.  $\det(A) \neq 0$  iff the columns of  $A$  are independent.
- (d) Adjugate matrix. Fact that  $A \cdot \text{adj}(A) = \det(A) \cdot I$ , hence  $A^{-1} = (1/\det(A))\text{adj}(A)$  when  $A$  is invertible.
- (e) Further properties:  $\det(AB) = \det(A)\det(B)$ , the determinant can be computed by Gaussian elimination, the determinant of a block triangular matrix is the product of the determinants of the blocks, if  $T(\mathbf{x}) = A\mathbf{x}$ , then the determinant of  $A$  measures the “volume expansion” associated with  $T$ .
- (f) “Correct” definition: the determinant is the unique alternating multilinear function  $d$  of  $n$  variables defined on  $\mathbb{R}^n$  for which  $d(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$ .
- (g) Permutation expansion of the determinant. Fact that  $\det(A) = \det(A^T)$ .
- (h) Cramer’s Rule for solving a linear system  $A\mathbf{x} = \mathbf{b}$  with invertible  $A$ .

### X. Abstract vector spaces.

- (a) Meaning of the word “abstract”.
- (b) Definition and examples of abstract vector spaces, e.g.,  $M_{m \times n}(\mathbb{R})$ ,  $\mathbb{P}_n(t)$ ,  $C^k([0, 1])$ . We computed that  $M_{m \times n}(\mathbb{R})$  has dimension  $mn$ ,  $\mathbb{P}_n(t)$  has dimension  $n + 1$ , and that  $C^0([0, 1])$  must be infinite dimensional.
- (c) Coordinates relative to a basis.
- (d) Definition of “isomorphism” of vector spaces. Proof that every finitely generated real vector space is isomorphic to  $\mathbb{R}^n$  for some finite  $n$ .
- (e) Matrices,  ${}_C[T]_B$ , for linear transformations between abstract vector spaces. Change of basis matrices,  ${}_C[I]_B$ .

### XI. Markov chains.

- (a) Definitions of: probability vector, (left, right, doubly) stochastic matrix and Markov chain.
- (b) Steady state vector.
- (c) Regular stochastic matrices have a unique steady state vector that is a probability vector.

### XII. Eigenvalues, eigenvectors, eigenspaces.

- (a) Eigenvectors identify “preserved directions” of a linear transformation  $T: V \rightarrow V$ .
- (b) Definitions of eigenvector, eigenvalue, eigenspaces.
- (c) Methods of calculation: characteristic polynomial  $\chi_A(\lambda)$  equals  $\det(A - \lambda I)$ ; e-values of  $A$  are the roots of  $\chi_A(\lambda) = 0$ ; e-space  $V_\lambda$  equals  $\text{Nul}(A - \lambda I)$ ;  $\lambda$ -eigenvectors are the nonzero vectors of  $V_\lambda$ . Fast calculation of e-values for (block) triangular matrices.

### XIII. Diagonalization.

- (a) Structure of roots of a real polynomial over  $\mathbb{R}$  or  $\mathbb{C}$ , and of a complex polynomial over  $\mathbb{C}$ . Algebraic multiplicity of an e-value.
- (b) Geometric multiplicity of an e-value.
- (c) Defn. of “diagonalizable”. Thm. A transformation  $T: V \rightarrow V$  is diagonalizable iff  $V$  has a basis consisting of e-vectors for  $T$  iff the geometric multiplicity of each e-value equals its algebraic multiplicity.
- (d) Independence of subspaces. Sums of subspaces and direct sums of independent subspaces. A sum of distinct e-spaces is direct.  $T: V \rightarrow V$  is diagonalizable iff  $V = \bigoplus_\lambda V_\lambda$ . (Side observation:  $\dim(U \oplus W) = \dim(U) + \dim(W)$ .)
- (e) Similarity:  $A$  is similar to  $B$  if  $A$  is a conjugate of  $B$ , i.e.,  $A = S^{-1}BS$ . Similarity is an equivalence relation on the set of  $n \times n$  matrices. Matrices are similar iff they represent the same transformation relative to different bases. Similar matrices have the same characteristic polynomial, hence same e-values. If  $A = S^{-1}BS$ , then  $S: V_\lambda^A \rightarrow V_\lambda^B$  is an isomorphism for each e-value  $\lambda$ .  $A$  is diagonalizable iff it is similar to a diagonal matrix.
- (f) (The nondiagonalizable case.) Generalized e-spaces  $V_\lambda^{(\infty)} = \bigcup_k \text{Nul}(A - \lambda I)^k$ .
- (g) The algebraic multiplicity of  $\lambda$  is  $\dim(V_\lambda^{(\infty)})$ .
- (h) If  $T: V \rightarrow V$  is defined over  $\mathbb{C}$  and  $V$  is f.g., then  $V$  is the direct sum of its generalized e-spaces.
- (i) Jordan block, Jordan canonical form. Eigenchains in  $V_\lambda^{(\infty)}$  yield JCF. Every transformation defined over  $\mathbb{C}$  is similar to a matrix in JCF, and the JCF is unique up to a permutation of Jordan blocks.
- (j) The JCF of  $A$  can be determined indirectly from the numbers ‘nullity( $A - \lambda I$ ) <sup>$k$</sup> ’ for all  $\lambda$  and  $k$ .
- (k) Diagonalization and JCF of  $T$  over  $\mathbb{R}$  instead of  $\mathbb{C}$ :  $\overline{V_\lambda^{(\infty)}} = V_\lambda^{(\infty)}$ , and  $V_\lambda^{(\infty)} \oplus \overline{V_\lambda^{(\infty)}}$  has a nice real basis consisting of the real and imaginary parts of the vectors in the  $\lambda$ -eigenchains in  $V_\lambda^{(\infty)}$ . This choice of basis results in diagonal form or JCF with  $2 \times 2$  real blocks replacing pairs of  $1 \times 1$  conjugate complex blocks.

### XIV. Orthogonality.

- (a) Dot product. (Defn. Arithmetic facts follow from those of matrices, since  $\mathbf{u} \bullet \mathbf{v} = \mathbf{u}^T \mathbf{v}$ .)
- (b) Length in  $\mathbb{R}^n$ . Unit vector in direction  $\mathbf{v}$  is  $\mathbf{v}/\|\mathbf{v}\|$ .
- (c) Angle in  $\mathbb{R}^n$  via  $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$ . Cauchy-Schwarz Inequality.
- (d) Orthogonality. Orthogonal complement.  $\text{Row}(A)^\perp = \text{Nul}(A)$ . Algorithm for finding the orthogonal complement of a set of vectors.
- (e) Orthonormal set of vectors. Angle-preserving linear transformations. Orthogonal matrices.
- (f) Approximate solutions to  $A\mathbf{x} = \mathbf{b}$  via least squares. Normal equations  $A^T A \mathbf{x} = A^T \mathbf{b}$ . Fitting curves to data.
- (g) Orthogonal projection onto a vector or subspace. Gram-Schmidt algorithm.

### General advice on preparing for a math test.

Be prepared to demonstrate understanding in the following ways.

- (i) Know the definitions of new concepts, and the meanings of the definitions.

- (ii) Know the statements and meanings of the major theorems.
- (iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
- (iv) Know how to perform the different kinds of calculations discussed in class.
- (v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
- (vi) Know how to correct mistakes made on old HW.

### Sample Problems.

- (1) From the book: Supplementary problems for chapters 3–6 (excluding problems marked [M]).
- (2) Computational problems:
  - (a) Using homogeneous coordinates, find a matrix representation for the transformation that rotates the plane  $45^\circ$  around the point  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .
  - (b) Find bases for the null space, row space and column space of the  $3 \times 3$  matrix whose entries are all 1. What are the dimensions of these spaces?
  - (c) Put the numbers  $1, 2, \dots, 9$  into a  $3 \times 3$  matrix in order. What is the determinant?
  - (d) Find a change of basis matrix from  $\mathcal{B} = \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$  and
 
$$\mathcal{C} = \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right).$$
  - (e) Find the characteristic equation, e-values, and e-spaces of  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ . Find a matrix  $S$  that conjugates  $A$  into diagonal form.
  - (f) Find a basis for  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}^\perp$ .
- (3) Can any of the following exist? (If so, give an example, if not give a reason.)
  - (a) A vector space with an empty basis.
  - (b) A matrix of rank zero.
  - (c) A matrix with no determinant.
  - (d) A matrix with a zero dimensional eigenspace.
  - (e) An invertible matrix whose row sums are all zero.
  - (f) A real matrix whose null space equals its column space.
  - (g) A matrix  $A$  such that  $\text{nullity}(A) = 1$  and  $\text{nullity}(A^2) = 3$ .
  - (h) A matrix where the dimension of the row space is greater than the dimension of the column space.
  - (i) A real number that does not arise as the determinant of a real matrix.
  - (j) A vector space with no subspaces.
  - (k) An isomorphism between vector spaces of different dimensions.
  - (l) A matrix whose row space is isomorphic to its column space.
  - (m) A matrix whose characteristic polynomial is  $\lambda^2 + \lambda + 1$ .
  - (n) An eigenvalue whose geometric multiplicity exceeds its algebraic multiplicity.
  - (o) A real  $10 \times 10$  matrix with only one eigenvector.
  - (p) A matrix equal to its adjugate.

- (q) A left stochastic matrix that is not right stochastic.
  - (r) Conjugate matrices of different ranks.
  - (s) Conjugate matrices that are not similar.
  - (t) A matrix that is diagonalizable over  $\mathbb{C}$  but not diagonalizable over  $\mathbb{R}$ .
  - (u) An orthogonal basis for  $\mathbb{R}^3$  that is not orthonormal.
  - (v) A nondiagonalizable complex matrix.
  - (w) An orthogonal matrix with determinant zero.
  - (x) A  $3 \times 3$  orthogonal matrix with no zero entries.
  - (y) Subspaces  $U$  and  $W$  such that  $U + W \neq U \oplus W$ .
  - (z) A real vector that is orthogonal to itself.
- (3) Give the dimensions of the following real vector spaces.
- (a) The space of real polynomials  $p(t)$  of degree at most 3 which satisfy  $p(1) = p(-1) = 0$ .
  - (b) The space of  $3 \times 3$  upper triangular real matrices.
  - (c) The space of twice continuously differentiable functions  $y = f(x)$  satisfying  $y'' = 0$ .
- (4) How would you solve the following problem? Suppose that  $V$  has basis  $(\mathbf{v}_1, \dots, \mathbf{v}_n)$  and  $U$  is a subspace of  $V$  with basis  $(\mathbf{u}_1, \dots, \mathbf{u}_m)$ . How do you find a basis for  $V$  whose first  $m$  vectors form a basis for  $U$ ?
- (5) Suppose that you are given bases  $\mathcal{B}$  and  $\mathcal{C}$  for subspaces  $U$  and  $W$  of a space  $V$ . How would you find a basis for  $U + W$ ? How would you find a basis for  $U \cap W$ ? (Hint: in both cases, you should apply Gaussian Elimination to the matrix  $[\mathcal{B}|\mathcal{C}]$ . How should you use the results?)
- (6) Is there a  $3 \times 3$  matrix whose minors are nonzero and all equal? Is there a  $3 \times 3$  matrix whose cofactors are nonzero and all equal?
- (7) Let  $S$  be a  $2 \times 2$  invertible matrix. Consider the linear transformation of “conjugation by  $S$ ”:

$$T: M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R}): A \mapsto S^{-1}AS.$$

Show that if  $\lambda$  is an e-value of  $T$ , then so is  $\lambda^k$  for any  $k$ . Show that 0 is not an e-value of  $T$ . Explain why the e-values of  $T$  can only be  $+1$  or  $-1$ . Show that  $+1$  occurs as an e-value with multiplicity at least 2.

- (8) What is the characteristic polynomial for the  $n \times n$  matrix whose entries are all 1?
- (9) Show that  $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$ . (Solution 1 hint: choose a basis for  $U \cap W$  and extend it in different ways to bases for both  $U$  and  $W$ . Show that all the vectors together form a basis for  $U + W$ .) (Solution 2 hint: let  $\mathcal{B}$  and  $\mathcal{C}$  be bases for  $U$  and  $W$ . Apply the rank+nullity theorem to the matrix  $[\mathcal{B}|\mathcal{C}]$ .)
- (10) The points  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ , and  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$  are the vertices of a regular tetrahedron. Find the lengths of the sides and the angles formed by adjacent faces.
- (11) Find the least squares curve of the form  $y = ax^2 + bx + c$  that best fits the data points  $(-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2)$ .
- (12) Show that if  $V$  is finite dimensional and  $U$  is a subspace, then  $V = U \oplus U^\perp$ .
- (13) Show that  $(U + W)^\perp = U^\perp \cap W^\perp$ .
- (14) Use the Gram-Schmidt process to find an orthonormal basis for the subspace of  $\mathbb{R}^3$  spanned by  $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$  and  $\begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$ .