LINEAR ALGEBRA (MATH 3130): REVIEW SHEET

I. Systems of linear equations.
  (a) Augmented matrix and coefficient matrix of a system.
  (b) Row reduction. (Reduced) row echelon form. Pivots, pivot positions, pivot columns.
  (c) Free and basic(=pivot) variables. Solutions sets. Parametrized form of a solution.
  (d) Consistent and inconsistent systems.
  (e) Homogeneous systems. Relationship between solutions of $A\mathbf{x} = \mathbf{b}$ and solutions of $A\mathbf{x} = \mathbf{0}$.

II. Matrix arithmetic.
  (a) Matrices can be added, negated, multiplied with each other, and scaled, provided the dimensions are right.
  (b) The collection of $n \times n$ real matrices forms an "$\mathbb{R}$-algebra", which is noncommutative if $n > 1$. (This statement indicates which laws of arithmetic are valid for $n \times n$ natrices, namely the ring laws. These laws are enumerated in Theorems 1 and 2 on pages 93 and 97.)
  (d) Matrix transpose.
  (e) Left and right inverses. (Equivalent properties.) Two-sided inverses. A 1-sided invertible matrix is 2-sided invertible iff it is square.
  (f) Elementary matrices. Row reduction is expressible as left multiplication by a sequence of elementary matrices. Matrices A and B are row equivalent iff $A = LB$ for some invertible matrix L.
  (g) Algorithm for finding inverses.
  (h) Partitioned matrices. Block diagonal and block triangular matrices.

III. Vectors and vector spaces.
  (a) Linear systems may be viewed as vector equations.
  (b) Definition of vector space. Definition of subspace.
  (c) Geometric interpretation of vector space operations.
  (d) Definition of column space and nullspace of a matrix.
  (e) Spanning set of vectors.
  (f) Linearly (in)dependent set of vectors.

IV. Linear Transformations.
  (a) Definition. Fact that any linear transformation has the form $T(\mathbf{x}) = A\mathbf{x}$.
  (b) The problem of solving the linear system $A\mathbf{x} = \mathbf{b}$ may be viewed as the problem of finding a vector $\mathbf{x} \in T^{-1}(\mathbf{b})$ for $T(\mathbf{x}) = A\mathbf{x}$.
  (c) One-to-one and onto transformations.
  (d) Finding the standard matrix of a transformation.
  (e) Matrices for rotation and reflection in the plane.

V. Matrix factorization.
  (a) LU factorization.
  (b) Solving systems with an LU factorization.

VI. Applications.
  (a) Balancing a chemical reaction.

(b) Network flow.
(c) Predator-prey dynamics.
(d) Leontief economic model.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) Chapter 1 supplementary problems (excluding problems marked [**M**]).

(2) Chapter 2 supplementary problems (excluding problems marked [**M**]).

(3) Let A be a square matrix. Explain why if the columns of A are independent, then the columns of A^2 are independent.

(4) Show that if $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is independent, then $\{\mathbf{a}, \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}+\mathbf{c}\}$ is also independent.

(5) Explain why if A and B are $n \times n$ matrices satisfying $A\mathbf{x} = B\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.

(6) Use the definition of "linear transformation" to show that the composition of two linear transformations is a linear transformation.

(7) Among all $n \times n$ matrices whose entries are all either 0 or 1 what is the maximum possible number of 1's if the matrix is invertible? What is the minimum number of 1's if the matrix is invertible? For which values of n is it possible for the number of 1's to be equal to the number of 0's and still have the matrix invertible?

(8) Explain why if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a subset of $\mathbb{R}^n$ that is linearly independent and spans the space, then $k = n$.

(9) Which matrices commute with $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$? (Hint: set up a linear system.)

(10) Explain why the set of columns of an $n \times n$ invertible matrix spans $\mathbb{R}^n$. Then explain why this set of columns is independent.

(11) Explain why every $n \times n$ matrix M is expressible in exactly one way as a sum $M = S + A$ where S is symmetric and A is antisymmetric.

(12) Show that if A is invertible, then A has at most one LU factorization.

(13) Give an example of a square matrix with no LU factorization.