Practice about matrices for transformations

(1) (a) Draw each of the following bases for $\mathbb{R}^2$ on its own copy of the plane: $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$, $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$, and $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2) = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$.

(b) Find the $\mathcal{E}$-coordinates, $\mathcal{B}$-coordinates, and $\mathcal{C}$-coordinates for the vector $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(c) Find the change of basis matrix ${}_{\mathcal{C}}[I]_{\mathcal{B}}$ and verify that ${}_{\mathcal{C}}[I]_{\mathcal{B}}[\mathbf{u}]_{\mathcal{B}} = [\mathbf{u}]_{\mathcal{C}}$.

(2) $\mathcal{B} = (1, t, t^2, t^3)$ and $\mathcal{C} = (1, t-1, (t-1)^2, (t-1)^3)$ are ordered bases for $\mathbb{P}_3(t)$. Find the change of basis matrices ${}_{\mathcal{C}}[I]_{\mathcal{B}}$ and ${}_{\mathcal{B}}[I]_{\mathcal{C}}$.

(3) In this problem use the basis $\mathcal{B} = (1, t, t^2, \ldots, t^m)$ for the polynomial space $\mathbb{P}_m(t)$.

(a) Find the matrix for differentiation $D \colon \mathbb{P}_n(t) \to \mathbb{P}_{n-1}(t) \colon f(t) \mapsto f'(t)$.

(b) Find the matrix for integration $J \colon \mathbb{P}_n(t) \to \mathbb{P}_{n-1}(t) \colon f(t) \mapsto \int_0^t f(x)\, dx$.

(c) Use the results from parts (b) and (c) to show that $D \circ J = I$ (the identity transformation), but $J \circ D \neq I$.

(d) Find a basis for the kernel of JD.