Practice writing proofs

Below are some statements for you to prove about left and right inverses. Some hints are on the back.

Theorem 1. *Show that the following are equivalent statements about a matrix A.*

(1) *A has a left inverse. (That is, there is a matrix B such that $BA = I$.)*
(2) *There is a unique solution to $A\mathbf{x} = \mathbf{0}$.*
(3) *The reduced row echelon form of A has a pivot in every column.*

Proof. [(1) $\Rightarrow$ (2)]

[(2) $\Rightarrow$ (3)]

[(3) $\Rightarrow$ (1)] (Hard part!)

$\square$

Theorem 2. *Show that the following are equivalent statements about an $m \times n$ matrix A.*

(1) *A has a right inverse. (That is, there is a matrix B such that $AB = I$.)*
(2) *A solution to $A\mathbf{x} = \mathbf{b}$ exists for every $\mathbf{b} \in \mathbb{R}^m$.*
(3) *The reduced row echelon form of A has a pivot in every row.*

Proof. [(1) $\Rightarrow$ (2)]

[(2) $\Rightarrow$ (1)]

[(2) $\Rightarrow$ (3)] (Hard part!)

[(3) $\Rightarrow$ (2)]

$\square$

Hints for Theorem 1.

[(1) $\Rightarrow$ (2)] Left multiply $A\mathbf{x} = \mathbf{0}$ by B.

[(2) $\Rightarrow$ (3)] There is at least one solution to $A\mathbf{x} = \mathbf{0}$. If some column is not a pivot column, there would be infinitely many solutions.

[(3) $\Rightarrow$ (1)] Suppose that the RRE form is $R = LA$, where L is a product of elementary matrices. R looks like the identity matrix with possibly some zero rows appended below. Explain why there is a matrix S such that $SR = I$, then take $B = SL$.

Hints for Theorem 2.

[(1) $\Rightarrow$ (2)] Right multiply $AB = I$ by $\mathbf{b}$.

[(2) $\Rightarrow$ (1)] Let $\mathbf{b}_i$ be a solution to $A\mathbf{x} = \mathbf{e}_i$ for $i = 1, \dots, m$. Let $B = [\mathbf{b}_1 \ \dots \mathbf{b}_m]$.

[(2) $\Rightarrow$ (3)] (Contrapositive) Suppose that the RRE form is $R = LA$, where L is a product of elementary matrices. If the last row of R is zero, then $R\mathbf{x} = \mathbf{e}_m$ is not solvable. Hence $L^{-1}R\mathbf{x} = L^{-1}\mathbf{e}_m$ is not solvable. Hence $A\mathbf{x} = \mathbf{b}$ is not solvable for $\mathbf{b} = L^{-1}\mathbf{e}_m$.

[(3) $\Rightarrow$ (2)] The RRE form for $[A|\mathbf{b}]$ cannot have a pivot in the final column.

If you have extra time, you might try to prove more equivalent properties. For example, show that the properties in Theorem 1 are also equivalent to:

Theorem 1 (4) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

Theorem 1 (5) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ preserves independent sets. (This means, if $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent, then $T(S) = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\}$ is also linearly independent.)

Then show that the properties in Theorem 2 are also equivalent to:

Theorem 2 (4) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

Theorem 2 (5) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ preserves spanning sets.