Practice Problems About Linear Independence, Linear Transformations.

(0) Which is right: "dependent" or "dependant"?

(1) Show that the set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is linearly independent in $\mathbb{R}^n$, and that it spans $\mathbb{R}^n$.

(2) Give a set S of three vectors in $\mathbb{R}^3$ that is dependent, yet no one of the vectors in S is a multiple of any other vector in S.

(3) Describe geometrically the linear transformations that are represented by each of the three types of elementary matrices.

(4) Explain the formula $T_A \circ T_B(\mathbf{x}) = T_{AB}(\mathbf{x})$. What (if anything) is worth remembering about this formula?

(5) Define $R_{\mathbf{v}}(\theta)\colon \mathbb{R}^3 \to \mathbb{R}^3$ to be the linear transformation that rotates vectors in 3-dimensional space through the axis in the direction of the vector $\mathbf{v}$ by an angle of θ radians.[1] Find the standard matrices for $R_{\mathbf{e}_1}(\frac{\pi}{2}), R_{\mathbf{e}_2}(\frac{\pi}{2})$, and the composite $R_{\mathbf{e}_2}(\frac{\pi}{2}) \circ R_{\mathbf{e}_1}(\frac{\pi}{2})$. The last matrix is also a rotation matrix through some axis; can you determine which axis?

(6) $r_\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear transformation that reflects vectors in 2-dimensional space through the line that goes through the origin and makes an angle α with the x-axis.
   (a) Find the standard matrices for r_α, r_β and the composite $r_\beta \circ r_\alpha$.
   (b) Explain why $r_\beta \circ r_\alpha$ is a rotation of the plane. Through what angle does it rotate?

(7) We write $\mathbb{P}_n$ for the collection of all polynomials

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

whose coefficients are real numbers and whose degree is at most n.
   (a) Explain why $\mathbb{P}_n$ is a real vector space, and why $\mathbb{P}_{n-1}$ is a subspace.

[1] To make this more precise, take the phrase "rotate through $\mathbf{v}$ by an angle of θ" to mean: fix a righthanded coordinate system $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with $\mathbf{v} = \mathbf{v}_1$. Now a "rotation through $\mathbf{v}$ by angle θ" is a linear transformation that fixes $\mathbf{v}$ and rotates the $(\mathbf{v}_2, \mathbf{v}_3)$ plane by θ radians in the counterclockwise direction.

(b) Identify $\mathbb{P}_2$ with the vector space $\mathbb{R}^3$ with the correspondence

$$a_0 + a_1 x + a_2 x^2 \mapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

Under this correspondence, show that differentiation of polynomials corresponds to a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$. What is the standard matrix for this linear transformation?