History of Mathematical Ideas
Quiz 10: Collaborative

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1. Show that if the curve defined by $AX^2 + BXY + CY^2 + DXZ + EYZ + FZ^2 = 0$ lies entirely on some line ℓ and the curve has at least two points on it, then the curve is the full line ℓ.

By applying a projective transformation we may assume that the curve lies entirely on the X-axis (equation $Y = 0$) and contains the two points $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. For the curve to go through these points we must have $A = F = 0$ in its equation.

The assumption that the curve defined by $BXY + CY^2 + DXZ + EYZ = 0$ lies entirely on the line $Y = 0$ implies that $B = D = E = 0$. To see this, there are cases to consider:

(1) If $C = 0$, then the curve is defined by $(BX + EZ)Y + DXZ = 0$ and any choice for X and Z leads to a value for Y. Since all points on the curve must satisfy $Y = 0$ it must be that $D = 0$. But now the curve is defined by $Y(BX + EZ) = 0$, which is the union of the lines $Y = 0$ and $BX + EZ = 0$. If $B \neq 0$ or $E \neq 0$, then one can find a solution to $BX + EZ = 0$ that does not lie on the line $Y = 0$. Hence $B = D = E = 0$.

(2) If $C \neq 0$, then we can view the defining equation as a quadratic equation in Y: $CY^2 + (BX + EZ)Y + DXZ = 0$. Using the quadratic formula, we can show that the only way this quadratic forces $Y = 0$ for every value of X and Z is if $B = E = 0$ and $D = 0$.

Since the equation of the defining curve has been reduced to $CY^2 = 0$, it follows that every point on the line $Y = 0$ lies on the curve.