Diagonalization and Jordan Canonical Form

Basic facts. Let V be a finite dimensional vector space and let $T\colon V \to V$ be a linear transformation. Fix an ordered basis $\mathcal{C}$ for V and let $A = {}_{\mathcal{C}}[T]_{\mathcal{C}}$ be the matrix for T relative to $\mathcal{C}$.

(1) A is diagonalizable iff V has a basis consisting of e-vectors for T iff the minimal polynomial of A has distinct roots.

(2) If $\mathcal{B}$ is a basis of e-vectors for T and $S = {}_{\mathcal{C}}[I]_{\mathcal{B}}$ is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$, then ${}_{\mathcal{B}}[T]_{\mathcal{B}} = S^{-1}AS$ is diagonal and the diagonal entries of $S^{-1}AS$ are the e-values of T in the correct order.

In practice, $\mathcal{C}$ is the standard basis. In this case, the columns of S are just the e-vectors of T written in the standard basis. Hence *the diagonalizing matrix is any matrix whose columns are a basis of e-vectors of T.*

(3) V has a basis of e-vectors for T iff V is a direct sum of the e-spaces for T:

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}.$$

(4) Some matrices are not diagonalizable, for example any $n \times n$ "Jordan block" is nondiagonalizable if $n > 1$:

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & 1 & & 0 & 0 & 0 \\ 0 & 0 & \lambda & \ddots & 0 & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & & \lambda & 1 & 0 \\ 0 & 0 & 0 & & 0 & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda \end{bmatrix}$$

(5) Any matrix A can be "almost diagonalized", if the scalar field is the complex numbers: there is always a matrix S whose columns are "generalized" eigenvectors of A arranged in eigenchains, such that $S^{-1}AS = J$ is block diagonal with each block a Jordan block.

Any matrix that is block diagonal with each block a Jordan block is said to be in Jordan Canonical Form (JCF).

(6) $\mathbf{v}$ is a "generalized e-vector" for e-value λ if it is nonzero and $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$ for some k. (When $k = 1$, $\mathbf{v}$ is a genuine e-vector.) We will not discuss how to find the matrix S that puts A into Jordan form, because the computations are very complicated, except to say that its columns are generalized e-vectors selected and ordered in a special way. Nevertheless it is not too hard to identify the Jordan form of A. Namely, the Jordan form of A is completely determined by the nullities of the matrices $(A - \lambda I)^k$ as λ ranges over e-values of A and k ranges from 1 to n.

Examples.

(A) Find the JCF of the 9×9-matrix A whose only e-value is 0 and which satisfies the conditions $\text{nullity}(A) = 4$, $\text{nullity}(A^2) = 7$, $\text{nullity}(A^3) = 8$, $\text{nullity}(A^4) = 9$.

Solution: Up to a permutation of Jordan blocks, there is only one matrix in JCF that has the same e-values (e-value $\lambda = 0$ with multiplicity 9) and the same nullities of the powers $(A - \lambda I)^k$. Namely, the fact that 9×9-matrix A has only the e-value 0 implies that the JCF of A has all 9 entries on its diagonal equal to 0. Also, $\text{nullity}(A - 0I) = 4$ implies that there are 4 Jordan blocks. Each block of size greater than 1 contributes an increase in nullity between $(A - \lambda I)$ and $(A - \lambda I)^2$. This increase is $7 - 4 = 3$, so 3 of the 4 Jordan blocks must have size > 1. Continuing this reasoning leads to the conclusion that the JCF of A is

$$\begin{bmatrix} J_4(0) & 0 & 0 & 0 \\ 0 & J_2(0) & 0 & 0 \\ 0 & 0 & J_2(0) & 0 \\ 0 & 0 & 0 & J_1(0) \end{bmatrix}$$

(B) Find the JCF of the 9×9-matrix A whose only e-value is 5 and which satisfies the conditions $\text{nullity}(A - 5I) = 4$, $\text{nullity}(A - 5I)^2 = 7$, $\text{nullity}(A - 5I)^3 = 8$, $\text{nullity}(A - 5I)^4 = 9$.

Solution: This is essentially the same as the previous problem:

$$\begin{bmatrix} J_4(5) & 0 & 0 & 0 \\ 0 & J_2(5) & 0 & 0 \\ 0 & 0 & J_2(5) & 0 \\ 0 & 0 & 0 & J_1(5) \end{bmatrix}$$

(C) Find the JCF of the 16×16-matrix A whose only e-values are 1 and -2 and which satisfies the conditions $\text{nullity}(A - I) = 3$, $\text{nullity}(A - I)^2 = 6$, $\text{nullity}(A - I)^3 = 8$, $\text{nullity}(A - I)^4 = 10$, $\text{nullity}(A - I)^5 = 10$, $\text{nullity}(A + 2I) = 2$, $\text{nullity}(A + 2I)^2 = 3$, $\text{nullity}(A + 2I)^3 = 4$, $\text{nullity}(A + 2I)^4 = 5$, $\text{nullity}(A + 2I)^5 = 6$, $\text{nullity}(A + 2I)^6 = 6$.

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