Theory of Rings

Homework 5

Problems.

1. Show that if G is a finite nonabelian group, then any faithful representation of degree 2 is irreducible. Show that if G is finite, centerless and has a faithful representation of degree 2, then the centralizer of any element of G is cyclic. Derive from the latter statement that G has cyclic Sylow subgroups.

2. Let G be a finite group and let H be a subgroup. Let $\mathbb{F}$ be a field whose characteristic does not divide $|G|$. Show that if every irreducible $\mathbb{F}$-representation of H has degree $\leq d$ and $[G : H] = e$, then every irreducible $\mathbb{F}$-representation of G has degree $\leq de$. What can one deduce from this about the degrees of the irreducible complex representations of the dihedral groups?

3. Let G be a finite group satisfying
(i) $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and
(ii) $Z(G) \cong \mathbb{Z}_2$.
Show that this information already determines the character table of G. Conclude that the two nonabelian 8-element groups have the same character tables.

4. Let $\mathrm{Irr}(G)$ be the set of irreducible characters of the finite group G. If $\chi \in \mathrm{Irr}(G)$ is afforded by a representation ρ, let $\det(\chi) := \det \circ \rho$ denote the character obtained by composing the usual determinant function with ρ.

Calculate the maps $\det\colon \mathrm{Irr}(G) \to \mathrm{Irr}(G)$ for each of the nonabelian 8-element groups, and show that they are different.

The *natural representation* of a subgroup G of $\mathrm{GL}(V)$ is the inclusion homomorphism $G \subseteq \mathrm{GL}(V)$. G is *irreducible* if its natural representation is. A *rotation* of $\mathbb{R}^n$ is an orthogonal transformation of determinant 1. An *axis of rotation* is an eigenvector with eigenvalue 1. The natural representation of a group $G \subseteq \mathrm{GL}(n, \mathbb{R})$ may be viewed as a complex representation via the inclusion $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathrm{GL}(n, \mathbb{C})$.

5. Show that a rotation group of a platonic solid is irreducible as a subgroup of $\mathrm{GL}(3, \mathbb{C})$. (Hint: let g_1, g_2, and g_3 be nontrivial rotations whose axes pierce the center of a face, the center of an edge of that face, and a vertex incident to that edge, respectively. Observe that the axes of these rotations are independent and pairwise nonorthogonal. Deduce irreducibility from the existence of such rotations.)

6. You are to prove the statement: If G is finite, $N \lhd G$, $x \in G$, and $\overline{x} = xN \in G/N$, then $|C_G(x)| \geq |C_{G/N}(\overline{x})|$.

(a) Prove the statement using characters.

(b) Prove the statement without using characters.

Assignment.

Group 1. (Hartman, Shannon) Problems 1 & 2.

Group 2. (Andrews, Bridges) Problems 3 & 5.

Group 3. (Blakestad, Havasi) Problems 4 & 6.