Theory of Rings

Homework 4

Read pages 79-111.

Problems.

1. Let $\mathbb{F}$ be a field, and let $\mathbf{A}$ be a finite dimensional $\mathbb{F}$-algebra. The *trace form* on $\mathbf{A}$ is the bilinear form $\mathrm{Tr}(xy)$, where $\mathrm{Tr}\colon \mathbf{A} \to \mathbb{F}$ is the trace functional.

(a) Prove that if the trace form is nondegenerate, then $\mathbf{A}$ is semisimple. (Hint: Prove that if a left ideal L is nil, then the trace functional vanishes on L.)

(b) Prove that if the trace form is degenerate and $\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F}) > \dim_{\mathbb{F}}(\mathbf{A})$, then $\mathbf{A}$ is not semisimple. (Hint: Show that the proof of (a) can be reversed when the characteristic is right. The characteristic restriction is needed to prove the converse of the hint for (a).)

Parts (a) and (b) prove that if the characteristic of $\mathbb{F}$ is right, then $\mathbf{A}$ is semisimple iff its trace form is nondegenerate. One way to establish nondegeneracy of the trace form is to prove that the matrix of the form is nonsingular. That matrix, computed relative to an ordered basis $(v_1, \ldots, v_n)$, is $[\mathrm{Tr}(v_i v_j)]$. *Using this method*, prove the following statements.

(c) Maschke's Theorem: If $\mathbb{F}$ is a field, G is a finite group, and $\mathrm{char}(\mathbb{F})$ does not divide $|G|$, then $\mathbb{F}[G]$ is semisimple.

(d) $M_n(\mathbb{F})$ is semisimple if field $\mathbb{F}$ has sufficiently nice characteristic. (Try to make "sufficiently nice" encompass as many characteristics as the method allows.)

2. Let G be a group, $\mathbb{F}$ a field, and $\sigma\colon G \to \mathrm{Aut}(\mathbb{F})$ a homomorphism. Discover and prove conditions on $G, \mathbb{F}$ and σ that are necessary and sufficient for the skew group ring $\mathbb{F}[G; \sigma]$ to be semisimple.

3. Let $\mathbb{F}$ be a field and G a group. Under what circumstances is the augmentation ideal of $\mathbb{F}[G]$ a nil ideal? When is it a nilpotent ideal?

Assignment.

All Groups. Read all exercises from Sections 3-5.

Group 1. (Andrews, Shannon) Problem 1 and Exercise 6.1 from Lam.

Group 2. (Blakestad, Hartman) Problem 2 from above and Exercise 6.2 from Lam.

Group 3. (Bridges, Havasi) Problem 3 from above and Exercise 6.14 from Lam.