# Theory of Rings

Homework 3

Read pages 63-78.

## Problems.

**1.** An ideal $P \subseteq R$ is *prime* if whenever $I$ and $J$ are ideals satisfying $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$. The intersection of all prime ideals is called the *prime radical* of $R$. Show that the prime radical of a ring is a nil ideal, hence is contained in $\mathrm{rad}(R)$. (Hint: show that if $r \in R$ is not nilpotent, then there is a prime ideal of $R$ not containing $r$.)

**2.** Show that any left noetherian ring has a nilpotent ideal that contains all nil left or right ideals. Hints:

(a) Prove that the ideal generated by a nilpotent left or right ideal is a nilpotent ideal.
(b) Prove that $Ra$ is nil iff $aR$ is nil.
(c) Prove that if $aR$ is nil and $b \in aR$ is a nonzero element such that $\mathrm{ann}_\ell(b)$ is maximal among left annihilators of nonzero elements, then $(bR)^2 = 0$. In fact, prove $bRb = 0$, as follows: Show that if $r \in R$, then either $br = 0$ or $\mathrm{ann}_\ell(br) = \mathrm{ann}_\ell(b)$. In either case, show that $brb = 0$.
(d) Deduce that if $R$ contains a nonzero nil left or right ideal, then $R$ contains a nonzero nilpotent ideal.
(e) Derive the result.

**3.**

(a) Let $R$ be a ring and $e \in R$ an idempotent. Show that $\mathrm{rad}(eRe) = e(\mathrm{rad}(R))e$.
(b) Show that $\mathrm{rad}(M_n(R)) = M_n(\mathrm{rad}(R))$.

## Assignment.

All Groups. Read all exercises from Sections 3-5.

Group 1. (Hartman, Moore) Problem 1 and Exercise 4.8 from Lam.

Group 2. (Andrews, Havasi) Problem 2 from above and Exercise 4.4 from Lam.

Group 3. (Blakestad, Bridges, Shannon) Problem 3 from above and Exercises 4.21 and 5.6 from Lam.