Theory of Rings

Homework 2

Read pages 30-63.

Problems.

1. Let V be an infinite dimensional D-vector space.

(a) Show that if I is a 2-sided ideal of $R := \mathrm{End}_D(V)$ and $r \in I \setminus \{0\}$ has infinite rank λ, then I contains every element of R whose rank is $\leq \lambda$.

(b) Show that the nonzero 2-sided of ideals of R are of the following two types. Any nonzero ideal is of the form

(i) $I_{<\lambda} := \{e \in \mathrm{End}_D(V) \mid \mathrm{rank}(e) < \lambda\}$, or

(ii) $I_{\leq\lambda} := \{e \in \mathrm{End}_D(V) \mid \mathrm{rank}(e) \leq \lambda\}$

for some infinite cardinal $\lambda \leq \dim_D(V)$.

2. Give an example of a ring R whose category of left modules is not equivalent to its category of right modules.

3. Show that a left artinian ring has finitely many isomorphism types of simple modules. What about a left noetherian ring?

4. Let V be a finite dimensional complex vector space. A linear transformation $T \in \mathrm{End}_{\mathbb{C}}(V)$ is called *semisimple* if it "acts semisimply on V", by which we mean that V is a semisimple module under the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$ that is generated by T. Show that T is semisimple iff it is diagonalizable.

Assignment.

All Groups. Read all exercises from Section 2.

Group 1. (Andrews, Bridges, Hartman) Problems 1, 2 from above and Exercise 3.1 (+ converse) from Lam.

Group 2. (Blakestad, Moore) Problem 3 from above and Exercise 3.5 from Lam.

Group 3. (Havasi, Shannon) Problem 4 and Exercise 3.6 (A& B) from Lam.