Theory of Rings

Homework 1

Read pages 1-30.

Problems.

1. Prove a Cayley-type theorem which faithfully represents an arbitrary small category as a subcategory of the category of sets. (Thus, any small category is isomorphic to a category whose objects are true sets and whose morphisms are true functions.) [Hint: let the functor map an object A of the small category $\mathcal{C}$ to the set of morphisms of $\mathcal{C}$ that have codomain A.]

2. (A characterization of matrix rings.) If R is a ring, then a subset

$$\{e_{ij} \mid 1 \leq i, j \leq n\} \subseteq R$$

is called a set of *matrix units* if it satisfies

(a) $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$, where δ_{jk} is the Kronecker delta function. ($\delta_{jk} = 1$ if $j = k$, $\delta_{jk} = 0$ if $j \neq k$.)

(b) $\sum_{i=1}^{n} e_{ii} = 1$.

(Note that matrix 'units' are not invertible in R if $n > 1$.)

(i) Let $R = M_n(T)$ for some ring T, and define $E_{ij} \in R$ to be the matrix whose ij-entry is 1 and whose other entries are 0, e.g.

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Show that $\{E_{ij} \mid 1 \leq i, j \leq n\}$ is a set of matrix units for R.

(ii) Show that if a ring S has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$, then $S \cong M_n(S')$ for some subring S' of S.

(iii) Show that a homomorphic image of $M_n(S)$ is isomorphic to $M_n(S/I)$ for some ideal I of S.

3. (One-sided ideals of $\mathrm{End}_{\mathbb{D}}(V)$.) Let M be an R-module, and let $S \subseteq M$ be a subset. The *annihilator* of S is $ann(S) = \{r \in R \mid rS = \{0\}\}$.

Let V be a finite dimensional (left) $\mathbb{D}$-vector space. The purpose of this exercise is to determine the left and right ideals of $R = \mathrm{End}_{\mathbb{D}}(V)$.

(i) Show that $L = ann(U)$ is a left ideal of R for any subspace $U < V$.

(ii) Show that if L is a left ideal of R, then $L = ann(U)$ for some subspace $U < V$. [Hint: Show that if $e, f \in R$ and $\ker(e) \not\supseteq \ker(f)$, then there is a $d \in R$ such that $\ker(de + f)$ is properly contained in $\ker(f)$. Conclude that if L is a left ideal and $f \in L$ has kernel of minimal dimension, then $L = R \cdot f = ann(\ker(f))$.]

(iii) Determine a similar correspondence between right ideals of R and subspaces of V.

(iv) Can anything interesting be said about the case where V is infinite dimensional?

4. (The matrix power functor.) Let R be a ring, $\varphi\colon M \to N$ be a homomorphism of R-modules, and n be a positive integer.

(i) Show that M^n is an $M_n(R)$-module, where the action of the ring on the module is that of multiplication of an $n \times n$ matrix by a column of length n.

(ii) Show that the map

$$\varphi^n\colon M^n \to N^n\colon \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \mapsto \begin{bmatrix} \varphi(m_1) \\ \vdots \\ \varphi(m_n) \end{bmatrix}$$

of φ acting coordinatewise is an $M_n(R)$-module homomorphism.

(iii) Show that $M \mapsto M^n$, $\varphi \mapsto \varphi^n$ is a functor from ${}_R\mathit{Mod}$ to ${}_{M_n(R)}\mathit{Mod}$.

5. (The localization functor.) An element $e \in R$ is *idempotent* if it satisfies $e^2 = e$. If e is an idempotent of R, then $eRe = \{ere \mid r \in R\}$ is a ring under the operations $\cdot, +, -, 0$ and e (as multiplicative identity element). Moreover, if M is an R-module, then $eM = \{em \mid m \in M\}$ is an eRe-module under the operations inherited from M. If $\varphi\colon M \to N$ is an R-module homomorphism, then the restriction $\varphi|_{eM}\colon eM \to eN$ is an eRe-module homomorphism.

(i) Show that $M \to eM$, $\varphi \mapsto \varphi|_{eM}$ is a functor from ${}_R\mathit{Mod}$ to ${}_{eRe}\mathit{Mod}$.

(ii) Show that this functor maps a simple R-module either to a zero eRe-module or to a simple eRe-module.

(iii) Show that if the ring R is semisimple, then so is eRe.

6. (Restriction of scalars.) Let $\varphi\colon R \to S$ be a ring homomorphism.

(i) If $\rho\colon S \to \mathit{End}_{\mathbb{Z}}(M)$ is the structure map for an S-module, then the composite $\rho \circ \varphi\colon R \to S \to \mathit{End}_{\mathbb{Z}}(M)$ is the structure map for an R-module. This defines an assignment ${}_SM \mapsto {}_RM$. Show that this is the object part of a functor from ${}_S\mathit{Mod}$ to ${}_R\mathit{Mod}$ (called ‘restriction of scalars’).

(ii) Since restriction of scalars is a functor, it takes isomorphic S-modules to isomorphic R-modules. Show that if φ is an epimorphism of rings, then restriction of scalars takes nonisomorphic S-modules to nonisomorphic R-modules.

Assignment.

Group 1. (Andrews, Blakestad) Problems 1, 2 from above and Exercise 1.14 from Lam.

Group 2. (Bridges, Gern) Problem 3 from above and Exercises 1.12 and 1.19 from Lam.

Group 3. (Hartman, Havasi) Problem 4 from above and Exercises 1.15(a)(b) and 1.16 from Lam.

Group 4. (A. Moore, Shannon) Problems 5, 6 from above and Exercise 1.7 from Lam.