Basic Properties of Characters of Finite Groups.

In this note, G is a finite group, V is an $\mathbb{C}[G]$-module, $\rho_V : G \to \mathrm{GL}(V)$ is the corresponding representation and $\chi_V = \mathtt{tr} \circ \rho_V$ is its character. (The subscripts will be omitted if they are irrelevant.) $\mathrm{Irr}(G)$ is the set of all irreducible characters of G. $K_G(g)$ is the conjugacy class of g in G.

Elementary linear algebra.

(1) $\rho(g)$ is diagonalizable.
(2) $\chi_V(1) = \mathtt{tr}(I) = \dim_{\mathbb{C}}(V)$ is the *degree* of χ_V.
(3) $\chi(hgh^{-1}) = \chi(g)$ (χ is a class function).
(4) $\chi(g)$ is the sum of $\chi(1)$ $|G|$-th roots of unity.
(5) $\chi(g^{-1}) = \overline{\chi(g)}$.
(6) $|\chi_V(g)| \le \chi_V(1)$, with equality iff $\rho_V(g) = \omega I$ for some $|G|$-th root of unity ω.

Kernel and center.

(7) $K_\chi := \{g \in G \mid \chi(g) = \chi(1)\}$ is a normal subgroup of G. (K_χ is the kernel of the associated representation, so it is called the *kernel* of χ.)
(8) $Z_\chi := \{g \in G \mid |\chi(g)| = \chi(1)\}$ is a normal subgroup of G containing K_χ.
(9) A subset $K \subseteq G$ is a normal subgroup iff $K = K_\chi$ for some (not necessarily irreducible) character χ. Thus, the characters of G determine its normal subgroups, its normal subgroup lattice, indices $[H : K]$ between normal subgroups, and whether or not G is solvable.
(10) $[G, G] = \bigcap_{\chi \text{ linear}} K_\chi$.
(11) $Z(G) = \bigcap_{\chi \in \mathrm{Irr}(G)} Z_\chi$. In fact, if $N \lhd G$, then $(N : G) = \bigcap_{\substack{\chi \in \mathrm{Irr}(G) \\ N \subseteq K_\chi}} Z_\chi$. Thus, the characters of G inductively determine the ascending central series of G, hence whether or not G is nilpotent.
(12) Z_χ / K_χ is cyclic.
(13) $Z_\chi / K_\chi \subseteq Z(G/K_\chi)$, with equality if $\chi \in \mathrm{Irr}(G)$.

Constructions of G-modules, and their characters. The constant homomorphism $\rho_1 : G \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^* : g \mapsto 1$ defines the trivial G-module; its character χ_1 is called the *principal character*. The principal character satisfies $\chi_1(g) = 1$ for all g.

If G acts on the set X via the homomorphism $\rho : G \to S_X$, then it acts on the $\mathbb{C}$-space with basis X via the homomorphism $\widehat{\rho} : G \to \mathrm{GL}(|X|, \mathbb{C})$ defined by $\widehat{\rho}(g)(\sum_{x \in X} a_x x) = \sum_{x \in X} a_x \rho(g)(x)$. Such a $\widehat{\rho}$ is called a *permutation representation*. The special case where $X = G$ and G acts on X by left multiplication is called the *regular representation*.

(14) Let G act on X with permutation character π. Then $\pi(g)$ equals the number of elements of X fixed by g.

(15) The character of the regular representation is $\pi(g) = \begin{cases} |G| & \text{if } g = 1; \\ 0 & \text{otherwise.} \end{cases}$

(16) The regular representation is faithful.

(17) $\chi_{U\oplus V} = \chi_U + \chi_V$.

(18) $\chi_{U\otimes V} = \chi_U \chi_V$.

(19) $\chi_{U^*} = \overline{\chi_U}$.

(20) $\chi_{\mathrm{Hom}_{\mathbb{C}}(U,V)} = \overline{\chi_U}\chi_V$.

(21) $\dim_{\mathbb{C}}(V^G) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$.

Inner Product. Let $\alpha, \beta\colon G \to \mathbb{C}$ be functions. Write $\langle \alpha, \beta \rangle$ for $\frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}\beta(g)$, which is the usual hermitian inner product on $\mathbb{C}^G$ weighted by the factor $1/|G|$.

(22) $\langle \chi_V, \chi_U \rangle = \langle \chi_V \overline{\chi_U}, \chi_1 \rangle = \langle \chi_{\mathrm{Hom}_{\mathbb{C}}(U,V)}, \chi_1 \rangle = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}[G]}(U, V) = \langle \chi_U, \chi_V \rangle$.

(23) (Row Orthogonality) [or "First Orthogonality Relation"] If $\chi_i, \chi_j \in \mathrm{Irr}(G)$, then $\langle \chi_i, \chi_j \rangle = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_i(g)}\chi_j(g) = \delta_{ij}$.

(24) (Column Orthogonality) ["Second Orthogonality Relation"] If $g, h \in G$ are not conjugate, then $\frac{1}{|G|}\sum_{\chi\in\mathrm{Irr}(G)}\overline{\chi(g)}\chi(h) = 0$. Otherwise this sum is $1/|K_G(g)|$.

(25) χ is irreducible iff $\langle \chi, \chi \rangle = 1$.

(26) Let G act on X with permutation character π. The number of orbits is $\langle \pi, \chi_1 \rangle$.

(27) Let G act on X with permutation character π. The action is 2-transitive iff $\pi = \chi_1 + \chi$ for some $\chi \in \mathrm{Irr}(G) - \{\chi_1\}$.

Integrality Properties.

(28) $\chi(g)$ is an algebraic integer.

(29) If $\chi \in \mathrm{Irr}(G)$, then $k_i\chi(g_i)/\chi(1)$ is an algebraic integer.

(30) If $\chi \in \mathrm{Irr}(G)$ and $\gcd(\chi(1), k_i) = 1$, then $\chi(g_i)/\chi(1)$ is an algebraic integer.

(31) If $\chi(g_i)/\chi(1)$ is an algebraic integer, then either $g_i \in Z_\chi$ or $\chi(g_i) = 0$.

(32) If $\chi \in \mathrm{Irr}(G)$, then $\chi(1)$ divides $|G|$.
Strengthenings:
(i) $\chi(1) \mid [G : Z_\chi]$.
(ii) $\chi(1)^2 \leq [G : Z_\chi]$.
(iii) $\chi(1) \leq [G : A]$ if $A \leq G$ and $[A, A] \leq K_\chi$.
(iv) $\chi(1) \mid [G : A]$ if $A \lhd G$ and $[A, A] \leq K_\chi$.

Example 1. (A character table.) The character table of a finite group G is the concatenation of the function tables of the irreducible characters of G. The first natural attempt to write down such a table for the group $S_3(\cong D_3)$ would produce:

S_3	1	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	-1	1	1
χ_3	2	0	0	0	-1	-1

But since each character is a class function, columns corresponding to conjugate group elements are identical. A more compact representation of the same information results from identifying duplicate columns. To record the details of the identification, choose a list of representatives $1 = g_1, g_2, \ldots, g_r$ for the conjugacy classes. Now delete all columns except those indexed by these elements. Record the size $k_i := |K_G(g_i)|$ of each class above the class representative. In general, the result is the table on the left, while the table for S_3 is given on the right.

	1	k_2	$\cdots$	k_r
G	1	g_2	$\cdots$	g_r
χ_1	1	1	$\cdots$	1
χ_2	d_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	d_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

	1	3	2
S_3	1	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

It is a convention to let the first column be that of the conjugacy class $\{1\}$, and the first row to be that of the principal character. (On the left, $d_i = \chi_i(1)$ is the degree of χ_i.)