Additional notes #1.

Categories and Functors

The intuitive model of a category is a collection of mathematical structures (called *objects*) equipped with structure preserving mappings (called *morphisms*). The formal definition closely resembles our definition of "algebra"; in fact, a (small) category is a 2-sorted partial algebra.

Definition 1. A *category* is a structure

$$\mathcal{C} = \langle O, M; \circ, \mathrm{id}, \mathrm{dom}, \mathrm{cod}\rangle$$

where

(1) $\mathrm{Ob}(\mathcal{C}) = O$ is a class whose members are called *objects*,
(2) $\mathrm{Mor}(\mathcal{C}) = M$ is a class whose members are called *morphisms*,
(3) $\circ : M \times M \to M$ is a binary partial operation called *composition*,
(4) $\mathrm{id} : O \to M$ is a unary function assigning to each object $A \in O$ a morphism id_A called *the identity of* A,
(5) $\mathrm{dom}, \mathrm{cod} : M \to O$ are unary functions assigning to each morphism f objects called the *domain* and *codomain* of f respectively.

The laws defining categories are:

(1) $f \circ g$ exists if and only if $\mathrm{dom}(f) = \mathrm{cod}(g)$.
(2) Composition is associative when it is defined.
(3) $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$, $\mathrm{cod}(f \circ g) = \mathrm{cod}(f)$.
(4) If $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$, then $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$.
(5) $\mathrm{dom}(\mathrm{id}_A) = \mathrm{cod}(\mathrm{id}_A) = A$.

Terminology & Notation.

A category is *small* if M is a set. (This forces O to be a set, too.) A category that is not small is *large*.

$\mathrm{Hom}_{\mathcal{C}}(A, B)$ denotes the class of $f \in M$ for which $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) = B$. A category is *locally small* if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set for all $A, B \in O$.
(So small = locally small + small object class.)

Examples.

(1) If $\mathcal{V}$ is a variety of algebras, then $\mathcal{V}$ may be thought of as a category whose objects are the algebras in $\mathcal{V}$ and whose morphisms are the homomorphisms between members of $\mathcal{V}$.
(2) If $\langle P; \leq\rangle$ is a partially ordered set, then the elements of P may be thought of as the objects of a category whose morphisms are the arrows $a \to b$ where $a \leq b$ in P.

(3) If $\langle M; \circ, 1\rangle$ is a monoid, then M determines a one-object category
$$\mathcal{M} = \langle \{*\}, M; \circ, 1, \mathrm{dom}, \mathrm{cod}\rangle$$
where $\mathrm{dom}, \mathrm{cod} : M \to \{*\}$ are both the constant function.

Since the definition of a category is so close to that of an algebra, it is natural to try to compare categories with "homomorphisms". These are called "covariant functors".

Definition 2. A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a "structure preserving mapping" from $\mathcal{C}$ to $\mathcal{D}$. Precisely, F is a pair of mappings, both called F, between object classes and morphism classes, $F(\mathrm{Ob}(\mathcal{C})) \subseteq \mathrm{Ob}(\mathcal{D})$ and $F(\mathrm{Mor}(\mathcal{C})) \subseteq \mathrm{Mor}(\mathcal{D})$, where

(1) $F(f \circ g) = F(f) \circ F(g)$,
(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,
(3) $F(\mathrm{dom}(f)) = \mathrm{dom}(F(f))$, and
(4) $F(\mathrm{cod}(f)) = \mathrm{cod}(F(f))$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a "composition reversing mapping" from $\mathcal{C}$ to $\mathcal{D}$. That is, $F(\mathrm{Ob}(\mathcal{C})) \subseteq \mathrm{Ob}(\mathcal{D})$, $F(\mathrm{Mor}(\mathcal{C})) \subseteq \mathrm{Mor}(\mathcal{D})$, and

(1) $F(f \circ g) = F(g) \circ F(f)$,
(2) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,
(3) $F(\mathrm{dom}(f)) = \mathrm{cod}(F(f))$, and
(4) $F(\mathrm{cod}(f)) = \mathrm{dom}(F(f))$.

Examples.

(1) Let $\mathcal{C}$ be any category, and let $A \in \mathrm{Ob}(\mathcal{C})$ be any object in $\mathcal{C}$. The *covariant hom functor represented by* A is the covariant functor $F : \mathcal{C} \to \mathcal{SET}$ whose behavior on objects is $F(X) = \mathrm{Hom}_{\mathcal{C}}(A, X)$ and whose behavior on morphisms is $F(f) =$ left composition with f.
(2) The *contravariant hom functor represented by* A is the contravariant functor $F : \mathcal{C} \to \mathcal{SET}$ whose behavior on objects is $F(X) = \mathrm{Hom}_{\mathcal{C}}(X, A)$ and whose behavior on morphisms is $F(f) =$ right composition with f.
(3) If we consider posets P and Q to be categories (as earlier), then a function from P to Q is a covariant functor if it is order-preserving and is a contravariant functor if it is order-reversing.
(4) If we consider monoids M and N as one-object categories, then a covariant functor from M to N is a monoid homomorphism.

Exercise. Check that these are functors.