Problem 6
Homework 5

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Proposition. *If G is a finite group, $N \lhd G$, $x \in G$, and $\bar{x} = xN \in G/N$, then*

$$|C_G(x)| \geq |C_{G/N}(\bar{x})|.$$

Proof. We have the following equalities:

$$|C_G(x)| = \frac{|G|}{|\mathcal{O}_x|}$$

and

$$|C_{G/N}(x)| = \frac{|G/N|}{|\mathcal{O}_{\bar{x}}|} = \frac{|G|}{|N||\mathcal{O}_{\bar{x}}|},$$

where $\mathcal{O}_x$ is the orbit of x under conjugation. If y, y' are elements of the same coset yN, then $y^{-1}xy$ and $y'^{-1}xy'$ are in the same coset. Hence there could be as many as $|N|$ distinct lifts of the element $\bar{y}^{-1}\bar{x}\bar{y}$ to elements $y^{-1}xy$. Thus

$$|\mathcal{O}_x| \leq |N||\mathcal{O}_{\bar{x}}|,$$

hence

$$\frac{|G|}{|\mathcal{O}_x|} \geq \frac{|G|}{|N||\mathcal{O}_{\bar{x}}|}.$$

The claim follows by the above equalities. □

We include a second proof using characters.

Proof. By property (24) of the Basic Properties of Characters of Finite Groups, we have the sum $\sum_{\chi \in Irr(G)} \overline{\chi(g)}\chi(h)$ is equal to 0 if $g, h \in G$ are not conjugate and equal to $|C_G(g)|$ otherwise. Hence $\sum_{\chi \in Irr(G)} |\chi(g)|^2 = |C_G(g)|$ for each g in G. Since each irreducible character on G/N extends to an irreducible character on G, we have

$$\sum_{\chi \in Irr(G/N)} |\chi(\bar{g})|^2 \leq \sum_{\chi \in Irr(G)} |\chi(g)|^2,$$

as the right sum is simply the left sum possibly with the addition of more positive terms, corresponding to the irreducible characters of G which do not arise from characters of G/N. But this is exactly the inequality $|C_{G/N}(\bar{x})| \leq |C_G(x)|$.

$\square$