MATH 6250: Theory of Rings
Homework 5
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4. Let $\mathrm{Irr}(G)$ be the set of irreducible characters of the finite group G. If $\chi \in \mathrm{Irr}(G)$ is afforded by a representation ρ, let $\det(\chi) := \det \circ \rho$ denote the character obtained by composing the usual determinant function with ρ.

Calculate the maps $\det : \mathrm{Irr}(G) \to \mathrm{Irr}(G)$ for each of the nonabelian 8-element groups, and show that they are different.

Solution.

There are two nonabelian 8-element groups, D_8 and Q_8. We have to find their character tables and representations that afford the characters.

The group Q_8 has five conjugacy classes: the elements of the center, 1 and -1, are one element classes, while there are three 2-element conjugacy classes, $\{i, -i\}$, $\{j, -j\}$ and $\{k, -k\}$. Thus we must find five irreducible characters. The first, of course, is the trivial character. The element $i \in Q_8$ generates the four element subgroup $\{1, -1, i, -i\}$, this is normal, as it has index 2. The factor group $Q_8/\langle i \rangle$ is $\mathbb{Z}_2$, and it has two irreducible characters, the non-trivial being $\mathbb{Z}_2 \ni 0 \mapsto 1$, $\mathbb{Z}_2 \ni 1 \mapsto -1$. We can lift this character to an irreducible character of the group Q_8, and similarly, we get two more irreducible characters by taking the factor groups $Q_8/\langle j \rangle$ and $Q_8/\langle k \rangle$. Thus we have four different linear irreducible characters, and because the squares of the character degrees must add up to 8, the fifth character must have degree 2. We can fill out the remaining places in the character table using the column orthogonality property. Thus we have the following character table for Q_8:

	1	1	2	2	2
Q_8	1	-1	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Now we need to find representations for these characters. Representations affording linear characters assign 1×1 matrices to group elements and the only element of a matrix is the same as the value of the character, so we can consider linear characters to be also representations. For χ_5 we found the following representation:

Q_8	1	-1	i	j	k
ρ_5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
$\det \circ \rho_5$	1	1	1	1	1

The third row of the table contains the determinants of the matrix representations, thus we can see that $\det(\chi_5) = \chi_1$. If we apply the function $\det : \mathrm{Irr}(G) \to \mathrm{Irr}(G)$ to a linear character, then it will return the same character, because the determinant of a 1×1 matrix is the same as the only element of the matrix. Thus the det function on the irreducible characters of Q_8 is the following:

$\chi \in \mathrm{Irr}(Q_8)$	χ_1	χ_2	χ_3	χ_4	χ_5
$\det(\chi)$	χ_1	χ_2	χ_3	χ_4	χ_1

The group D_8 also has five conjugacy classes, the elements of the center, 1 and r^2, form one element conjugacy classes and the three 2-element conjugacy classes are $\{r, r^3\}$, $\{s, sr^2\}$ and $\{sr, sr^3\}$. If we factor D_8 by the center, then the four co-sets are $\{1, r^2\}$ and the three 2-element conjugacy classes, and the factor group is the Klein 4 group. We can find the character table of the Klein 4 group by factoring it by its 2-element subgroups and lifting the character table of $\mathbb{Z}_2$:

$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Now we can lift the character table of the Klein 4 group to obtain four irreducible characters of D_8. As in the case of Q_8, the fifth character of D_8 must have degree 2, and we can find its values using the column orthogonality property of the character table. We see that the character table of D_8 is identical to that of Q_8:

	1	1	2	2	2
D_8	1	r^2	r	s	sr
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

We can find a representation affording χ_5 using that D_8 contains symmetries of a square, so r is a 90 degree rotation and s is reflection about the x axis:

D_8	1	r^2	r	s	sr
ρ_5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
$\det \circ \rho_5$	1	1	1	-1	-1

From the third row containing the matrix determinants we see that $\det(\chi_5) = \chi_2$. Thus the function det is different for Q_8 and D_8: in the first case the degree two character is taken to the trivial character, while in the second case the degree two character is taken to a nontrivial linear character.