Problem 3 Scott Andrews, Clifford Bridges

Let G be a finite group satisfying

1. $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
2. $Z(G) \cong \mathbb{Z}_2$

Show that this information already determines the character table of G. Conclude that the two nonabelian 8-element groups have the same character tables.

As G is nonabelian, it has at least one irreducible character with degree greater than 1. The sum of the squares of the degrees of the irreducible characters of G must be 8, hence G has four linear characters and one character of degree 2. G must have five conjugacy classes, exactly two of which are singleton sets (composing the center). As the sizes of the conjugacy classes must divide 8, the other three conjugacy classes are all 2-element sets.

Let $1 = \chi_1$, χ_2, χ_3 and χ_4 be the linear characters of G and χ_5 be the degree 2 irreducible character of G. Let $\{1\} = K_1$ and K_2 be the 1-element conjugacy classes of G, and let K_3, K_4 and K_5 be the 2-element conjugacy classes of G.

There are four irreducible character of G lifted from $G/Z(G)$, each of which must be linear (as $G/Z(G)$ is abelian). The character table of $G/Z(G)$ is easily determined; in fact, every entry must be ± 1 as all elements have order 2. We now have a partial character table for G given by

	K_1	K_2	K_3	K_4	K_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	-1	1	-1
χ_4	1	1	1	-1	-1
χ_5	2				

(up to permutation of K_3, K_4 and K_5 and permutation of χ_2, χ_3 and χ_4). The remaining entries can be determined by the column orthogonality relation, and the character table of G is

	K_1	K_2	K_3	K_4	K_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	1	-1	1	-1
χ_4	1	1	1	-1	-1
χ_5	2	-2	0	0	0

It suffices to show that if G is a nonabelian 8-element group, then $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $Z(G) \cong \mathbb{Z}_2$.

Lemma. Let G be a group such that $G/Z(G)$ is cyclic; then G is abelian.

Proof. As $G/Z(G)$ is cyclic, we can write $G/Z(G) = \langle aZ(G) \rangle$ for some $a \in G$. Then

$$\begin{aligned} G/Z(G) &= \{(aZ(G))^k \mid k \in \mathbb{Z}\} \\ &= \{a^k Z(G) \mid k \in \mathbb{Z}\} \end{aligned}$$

It follows that each element of G can be written in the form $a^k z$ where $k \in \mathbb{Z}$ and $z \in Z(G)$. It is clear that all elements of this form commute, hence G is abelian.

$\square$

This lemma implies that $|Z(G)|$ cannot be 4 or 8. As G is a p-group, G has a nontrivial center, hence $|Z(G)| = 2$, and $Z(G) \cong \mathbb{Z}_2$. Then $|G/Z(G)| = 4$, and as $G/Z(G)$ is not cyclic by the lemma, $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.