Problem. Let G be a finite group and let H be a subgroup. Let $\mathbb{F}$ be a field whose characteristic does not divide $|G|$. Show that if every irreducible $\mathbb{F}$-representation of H has degree $\leq d$ and $[G : H] = e$, then every irreducible $\mathbb{F}$-representation of G has degree $\leq de$. What can one deduce from this about the degrees of the irreducible complex representations of the dihedral groups?

Solution. Let $\rho : G \to M_n(\mathbb{F})$ be an irreducible representation of G. If the restriction of ρ to H is also an irreducible representation of H, then ρ must have degree $\leq d \leq de$. If not, then $\rho|_H$ is not an irreducible representation of H.

Let $V = \mathbb{F}^n$ be the irreducible G-module corresponding to ρ. By restriction of scalars, V is also an H-module, although not irreducible. So $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ where the V_i are (not necessarily distinct) irreducible H-modules.

The sum

$$\sum_{g \in G} gV_1$$

is a G-submodule of V, hence must equal V since V is a simple G-module. As V_1 is an irreducible H-module, V_1 has dimension $\leq d$ as an H-module. H fixes each of the V_i setwise, so in particular $hV_1 = V_1$ for any $h \in H$. Now suppose that g and g' lie in the same left coset of H in G. Then $g = g'h$ for some $h \in H$, and hence

$$gV_1 = g'hV_1 = g'V_1.$$

So, letting $\{g_1, \ldots, g_e\}$ be a collection of coset representatives for H, we have

$$V = \sum_{j=1}^{e} g_j V_1.$$

Since each $g_j V_1$ is spanned by $\leq \dim(V_1) \leq d$ vectors, we have that V (as a G-module) is spanned by $\leq de$ vectors, and $\dim V \leq de$. So ρ has degree $\leq de$.

What can one deduce from this about the degrees of the irreducible complex representations of the dihedral groups?

Every dihedral group contains a subgroup R containing the identity and all rotations. The subgroup R is cyclic, hence abelian, and therefore every irreducible complex representation of R has degree one. The index of R in the dihedral group is two; therefore, any irreducible complex representation of the dihedral group must have degree at most two.