**Theory of Rings Homework**

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**Rings5p1:** Show that if $G$ is a finite nonabelian group, then any faithful representation of degree 2 is irreducible. Show that if $G$ is finite, centerless and has a faithful representation of degree 2, then the centralizer of any (non-identity) element of $G$ is cyclic. Derive from the latter statement that $G$ has cyclic Sylow subgroups.

**Proof:** (part 1) Let $\rho : G \to \mathrm{GL}_2 V$ be a faithful representation of $G$ degree 2. If $\rho$ is reducible, then the $G$-module $V$ decomposes as $V_1 \oplus V_2$ where $V_1$ and $V_2$ are 1-dimensional. Since $[G, G]$ acts trivially on 1-dimensional modules, $[G, G]$ acts trivially on $V_1 \oplus V_2 = V$. Since the action of $G$ on $V$ is faithful, this implies that $[G, G] = 1$, which contradicts that $G$ is nonabelian. Thus, $\rho$ must be irreducible.

(part 2) Let $G$ be a finite, centerless group of order $n$ and let $\rho : G \to \mathrm{GL}_2(V)$ be a faithful representation of degree 2. Choose a nonidentity element $g \in G$. Pick a basis for $V$ so that $\rho(g)$ is a diagonal matrix. Since $\rho$ is faithful and irreducible and $G$ is centerless, $\rho(g)$ is not in the center of $\mathrm{GL}_2(V)$. Hence the matrix for $\rho(g)$ is a nonscalar diagonal matrix, say $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where $\lambda_1$ and $\lambda_2$ are $n^{th}$ roots of unity. The homomorphism $\rho$ must map $C_G(g)$ into the centralizer of $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Hence the image under the restriction of $\rho$ to $C_G(g)$ is a subgroup of diagonal matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b$ are $n^{th}$ roots of unity. We claim that the image of $C_G(g)$ under $\rho$ is cyclic. Indeed, there is an isomorphism from $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} / \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \cong \begin{bmatrix} a/b & 0 \\ 0 & 1 \end{bmatrix}$. But there are no non-identity scalar diagonal matrices in the image of $\rho$ because $G$ is centerless. Thus $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cong \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} / \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \cong \begin{bmatrix} a/b & 0 \\ 0 & 1 \end{bmatrix}$ which is clearly cyclic because $a/b$ is an $n^{th}$ root of unity. Thus $C_G(g)$ maps injectively into a cyclic group, hence $C_G(g)$ is cyclic.

(part 3) Let $G$ be as in part 2. Given any Sylow $p$-subgroup $P$ of $G$, choose $g \in Z(P) \setminus \{1\}$. $P$ is a subgroup of the cyclic group $C_G(g)$, hence is itself cyclic.