Theory of Rings Homework

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Rings4p6.2:

Let A be a normal elementary p-subgroup of a finite group G such that the index of the centralizer $C_G(A)$ is prime to p. Show that for any normal subgroup B of G lying in A, there exists another normal subgroup C of G lying in A such that $A = B \times C$. (**Hint.** Consider the conjugation action of G on A and apply Maschke's Theorem with $k = \mathbb{F}_p$.)

In this proof, we will be using Maschke's Theorem over a field, which can be stated as: Let A be a vector space over a field k such that A is a G-module. If G is finite, and $|G|$ is a unit in k, then A is semisimple as a G-module.

Let k be the field $\mathbb{F}_p$, G be a finite group, and A be a normal elementary p-subgroup of G with $C_G(A)$ having index in G prime to p. Because A is an elementary p-subgroup, A is an abelian subgroup of G. Because A is abelian and normal, A is a G-module where G acts on A by conjugation. A is definitionally equivalent to a vector space over k by defining scalar multiplication in terms of addition as the group operation from G (e.g. $2x = x + x$). We claim that submodules of A correspond to normal subgroups of G contained in A. To show this, let B be a submodule of A. Then, B is contained in A, and B must be closed under the same G action on A, namely conjugation by G, and hence $B \trianglelefteq G$. Conversely, let $B \trianglelefteq G$ and $B \subseteq A$. Then B is closed under conjugation by G, and B is an abelian group because A is an abelian group. Thus, B is a submodule of A. The action of G on A induces a homomorphism $\rho : G \to \mathrm{GL}(A)$. The kernel of ρ is $C_G(A)$, so ρ factors as $G \overset{nat}{\to} G/C_G(A) \overset{\bar{\rho}}{\to} \mathrm{GL}(A)$. This shows that the action of G on A is induced by the restriction of scalars by the action of $G/C_G(A)$ on A. By Maschke's Theorem A is semisimple as a $G/C_G(A)$-module. Hence A is semisimple as a G-module. This shows that the lattice of G-invariant subspaces of A is complemented. Hence the lattice of normal subgroups of G lying in A is complemented. Thus, because B is a normal subgroup of G contained in A, there exists a normal subgroup C of G contained in A such that $A = B \times C$.