MATH 6250: Theory of Rings
Homework 4
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Ex. 6.13. (Passman) Assume *char* $k = 2$. Let A be an abelian $2'$-group and let G be the semidirect product of A and a cyclic group $\langle x \rangle$ of order 2, where x acts on A by $a \mapsto a^{-1}$.
(1) If $|A| < \infty$, show that $rad\, kG = k \cdot \sum_{g \in G} g$, and $(rad\, kG)^2 = 0$.
(2) If A is infinite, show that kG has no nonzero nil ideals.

Ex. 6.14. (Wallace) Assume *char* $k = 2$, and let $G = A \cdot \langle x \rangle$ as in Exercise 13, where A is the infinite cyclic group $\langle y \rangle$. (G is the infinite dihedral group.) Show that $R = kG$ is J-semisimple (even though G has an element of order 2).

Solution 6.13.

We notice first that kA has no nonzero nilpotent elements: it follows from Proposition 6.13 or Passman's Theorem (6.15) that kA has no nonzero nil ideals. Then, as kA is commutative, any nilpotent element would generate a nil ideal, therefore the only nilpotent element in kA is zero.

Any element in kG can be expressed in the form $\alpha + \beta x$, $\alpha, \beta \in kA$. For $\alpha = \sum_{a \in A} \alpha_a a \in kA$, let $\alpha^* = \sum \alpha_a a^{-1}$; then $x\alpha = \alpha^* x$. Let I be any nil ideal in kG and $\alpha + \beta x \in I$. Then $(\alpha + \beta x)(\alpha^* + x\beta^*) = \alpha\alpha^* + \alpha x \beta^* + \beta x \alpha^* + \beta x x \beta^* = \alpha\alpha^* + \alpha\beta x + \beta\alpha x + \beta\beta^* = \alpha\alpha^* + \beta\beta^*$, using that α and β commute and $char\, k = 2$. As $\alpha\alpha^* + \beta\beta^*$ is in the ideal I it is nilpotent. As it is also in kA, it has to be zero by our previous observation. As $char\, k = 2$, $\alpha\alpha^* + \beta\beta^* = 0$ means $\alpha\alpha^* = \beta\beta^*$. Next we show that $(\alpha + \beta x)^m = (\alpha + \alpha^*)^{m-1}(\alpha + \beta x)$. To see this it is enough to prove that $(\alpha + \beta x)(\alpha + \beta x) = (\alpha + \alpha^*)(\alpha + \beta x)$:

$$(\alpha + \beta x)(\alpha + \beta x) = \alpha\alpha + \alpha\beta x + \beta x \alpha + \beta x \beta x = \alpha\alpha + \alpha\beta x + \beta\alpha^* x + \beta\beta^* =$$
$$= \alpha\alpha + \alpha\beta x + \alpha^*\beta x + \alpha^*\alpha = (\alpha + \alpha^*)(\alpha + \beta x).$$

Now, as $\alpha + \beta x$ is nilpotent, $(\alpha + \alpha^*)^k(\alpha + \beta x) = 0$ for some integer k. But this is only possible if $\alpha + \alpha^* = 0$: suppose that $(\alpha + \alpha^*)^k(\alpha + \beta x) = 0$. Then $(\alpha + \alpha^*)^k\alpha = 0$ and also $((\alpha + \alpha^*)^k\alpha)^* = (\alpha + \alpha^*)^k\alpha^* = 0$, thus $(\alpha + \alpha^*)^{k+1} = 0$. As the only nilpotent element of kA is zero, we have that $\alpha + \alpha^* = 0$ and thus $\alpha = \alpha^*$. If $\alpha + \beta x \in I$ then $(\alpha + \beta x)x = \alpha x + \beta \in I$ and so $\beta = \beta^*$.

As $\alpha = \alpha^*$, we can write $\alpha = c \cdot 1 + \sum c_{a_i}(a_i + a_i^{-1})$, using that A is a $2'$-group and so the only element in A whose inverse is itself is the identity. Because I is an ideal, $a_i^{-1}(\alpha + \beta x) \in I$, and so $a_i^{-1}\alpha = c \cdot a_i^{-1} + \sum c_{a_i}(1 + a_i^{-2}) = (a_i^{-1}\alpha)^*$, and in $a_i^{-1}\alpha$ the coefficient of a_i^{-1} and a_i must be the same (c), which means that in α the coefficient of a_i^2 must also be c. This means that in α all the group elements of A that are squares must have the same coefficient as the identity element. Elements of A with finite order are squares, because A is a $2'$-group, and so if the order of $a \in A$ is $2k + 1$, then a is the square of a^{k+1}. Thus we proved that if $\alpha + \beta x \in I$, where I is a nil ideal, then in α all the elements of A that are squares (in particular elements of finite order) have the same coefficient as the identity element and the same statement is true for β.

Now, let's assume first that A is infinite. If A has an element of infinite order, then we have infinitely many squares in A, taking the even powers of the element of infinite order. On the other hand if all the elements of A have finite order, then again we have infinitely many squares in A, because all elements of finite order are squares. In any case we have infinitely many squares in A. Now let I be a nil ideal in kG and suppose that it contains a nonzero element $\alpha + \beta x \neq 0$, thus either $\alpha \neq 0$ or $\beta \neq 0$. If for example α is nonzero, then by multiplication we can have that the coefficient of the identity in $a\alpha$ is nonzero for some $a \in A$. Then $a(\alpha + \beta x) \in I$ and by our previous observations in $a\alpha$ infinitely many elements of A should have nonzero coefficient, because the coefficient of the identity is nonzero and there are infinitely many squares in A whose coefficients must be the same as the identity's coefficient. This is not possible and this proves part (2) of the statement.

To prove part (1), assume that A is finite and let $J = k \cdot \sum_{g\in G} g = k \cdot (g_1 + g_2 + \cdots + g_n)$. As multiplication by any $g_i \in G$ permutes the elements of G, we have that $g_i \sum_{g\in G} g = \sum_{g\in G} g$. Thus $(\sum_{g\in G} g)^2 = (\sum_{g\in G} g)(\sum_{g\in G} g) = g_1(\sum_{g\in G} g) + \cdots + g_n(\sum_{g\in G} g) = (\sum_{g\in G} g) + \cdots + (\sum_{g\in G} g) = n(\sum_{g\in G} g) = 0$, as $n = 2 \cdot |A|$ and $char\, k = 2$. This means that $J^2 = 0$ and $J \subseteq rad\, kG$. To prove that $rad\, kG = J$, assume that $\alpha + \beta x \in rad\, kG$, but $\alpha + \beta x \notin J$. As G is finite, kG is a finite dimensional k vector space, so it is artinian as a ring, which means that the Jacobson radical is nilpotent and thus nil. Thus $\alpha + \beta x \in rad\, kG$, a nil ideal, and by what we proved earlier, all elements in α must have the same coefficients and the same holds for β, so $\alpha = r \sum_{a\in A} a$ and $\beta = s \sum_{a\in A} a$ for some $r, s \in k$. On the other hand by our assumption $\alpha + \beta x \notin J$, so $\alpha \neq \beta$. As by the definition of J we have that $\beta + \beta x \in J \subseteq rad\, kG$, we have that $(\alpha + \beta x) - (\beta + \beta x) = \alpha - \beta \in rad\, kG$. But this is not possible, because kA does not have any nonzero nilpotent elements. Thus part (1) is proved.

Solution 6.14.

Let $H_i = \langle y^{3^i} \rangle$ $(1 \leq i < \infty)$. Then $H_i \lhd G$, and $G_i = G/H_i$ is the dihedral group of order $2 \cdot 3^i$, because a presentation of G is $\langle x, y \,|\, x^2 = 1, yx = xy^{-1} \rangle$, and factoring by H_i just adds $y^{3^i} = 1$ to the relations and the presentation becomes that of the finite dihedral group of order $2 \cdot 3^i$. Now we can define the ring homomorphism $\varphi : kG \to \prod_{i=1}^{\infty} kG_i$, sending $\sum \alpha_g g$ to $(\sum \alpha_g g H_1, \sum \alpha_g g H_2, \ldots)$. This map is injective: let $t = \sum \alpha_g g \neq 0$, where each g in the sum has the form y^j or $y^j x$. As there are only finitely many group elements in t with nonzero coefficients, the values of j has a maximum in t, and we can choose i such that 3^i is greater than the maximum of the j values. Then the ith component of $\varphi(t)$ is different from zero and thus φ is injective.

Now let $s, t \in rad\, kG$. Then because of Lemma 4.1 and the fact that each component φ_i of the map φ is surjective, we have that $\varphi_i(s), \varphi_i(t) \in rad\, kG_i$. From part (1) of Exercise 13 we know that $(rad\, kG_i)^2 = 0$, thus $\varphi_i(s)\varphi_i(t) = 0$ for all i. But then $\varphi(s)\varphi(t) = \varphi(st) = 0$ and because of the injectivity of φ, $st = 0$ and $(rad\, kG)^2 = 0$. From part (2) of Exercise 13 we know that kG has no nonzero nil ideals, thus $rad\, kG = 0$ and kG is J-semisimple.