Problem. Let V be a kG-module and H be a subgroup of G of finite index m not divisible by *char* k. Modify the proof of Maschke's Theorem to show the following: if V is semisimple as a kH-module, then V is semisimple as a kG-module.

Solution. Suppose that V is semisimple as a kH-module. We'll show that V is semisimple as a kG-module by showing that the kG-submodule lattice of V is complemented.

Let N be any nontrivial proper kG-submodule of V, and hence also a kH-submodule of V. Since V is semisimple as a kH-module, there exists a kH-submodule C in V such that $V = N \oplus C$. Therefore any element of V can be uniquely written as $n + c$ for some $n \in N$ and $c \in C$. We define a kH-module homomorphism $\varphi : V \to V$ by $\varphi(n + c) = n$. This φ is projection onto N, so $\varphi^2 = \varphi$ and $\varphi|_N = \mathrm{id}_N$.

Let $\{a_1, a_2, \ldots, a_m\}$ be a list of coset representatives for H. We now define a new map $\psi : V \to V$ by

$$\psi(v) = \frac{1}{m}\sum_{i=1}^{m}(a_i \cdot \varphi)(v) = \frac{1}{m}\sum_{i=1}^{m} a_i\varphi(a_i^{-1}v).$$

Claim. $\psi|_N = \mathrm{id}_N$.

If $n \in N$, then $a_i^{-1}n \in N$ for each i since N is a kG-submodule of G. So

$$\psi(n) = \frac{1}{m}\sum_{i=1}^{m} a_i\varphi(a_i^{-1}n) = \frac{1}{m}\sum_{i=1}^{m} a_i a_i^{-1}n = \frac{1}{m}\sum_{i=1}^{m} n = n. \quad \square$$

Claim. $\operatorname{Im}\psi = N$.

From above, we already know that $\operatorname{Im}\psi \supseteq N$.

If $v \in V$, then $\varphi(a_i v) \in N$ for each i since $\operatorname{im}\varphi = N$. Since N is a kG-submodule, $a_i\varphi(a_i^{-1}v) \in N$ for each i, and hence $\psi(v) \in N$ also. Thus $\operatorname{Im}\psi \subseteq N$, so $\operatorname{Im}\psi = N$. $\square$

The two claims above together show that $\psi^2 = \psi$.

Claim. If $x \equiv y \pmod{H}$, then $x\varphi(x^{-1}v) = y\varphi(y^{-1}v)$ for all $v \in V$.

Suppose $x \equiv y \pmod{H}$. Then $x = yh$ for some $h \in H$. So for any $v \in V$,

$$x\varphi(x^{-1}v) = (yh)\varphi((yh)^{-1}v) = yh\varphi(h^{-1}y^{-1}v) = yhh^{-1}\varphi(y^{-1}v) = y\varphi(y^{-1}v). \quad \square$$

This shows that our choice of coset representatives $\{a_1, \ldots, a_m\}$ for the construction of ψ does not matter, and another choice of coset representatives would have given the same map ψ.

Claim. ψ is a kG-module homomorphism.

Let $g \in G$ and $v \in V$. It suffices to show that $(g \cdot \psi)(v) = \psi(v)$. We compute:

$$\begin{aligned}
(g \cdot \psi)(v) &= g\psi(g^{-1}v) \\
&= g\frac{1}{m}\sum_{i=1}^{m} a_i\varphi(a_i^{-1}g^{-1}v) \\
&= \frac{1}{m}\sum_{i=1}^{m}(ga_i)\varphi((ga_i)^{-1}v) \qquad \text{since } \frac{1}{m} \in k \\
&= \psi(v) \qquad \text{since } \{ga_1, ga_2, \ldots, ga_m\} \text{ is another set of coset representatives for } H.
\end{aligned}$$
$\square$

Let $K = \ker\psi$.

Claim. K is a kG-module complement for N.

We know that K is a kG-submodule of V, since K is the kernel of a kG-module homomorphism.

To see that $V = K + N$, let $v \in V$. Write $n = \psi(v)$ and $k = v - \psi(v)$. We have $n \in N$ since $n \in \operatorname{Im}\psi = N$, and $k \in K$ since $\psi(k) = \psi(v) - \psi^2(v) = 0$. And $n + k = \psi(v) + v - \psi(v) = v$. So $v \in N + K$, and therefore $V = N + K$.

To see that $K \cap N = \{0\}$, suppose that $v \in K \cap N$. Then $\psi(v) = v$ since $v \in N$. But $\psi(v) = 0$ since $v \in K$. So $v = 0$. $\square$

Therefore K is a kG-module complement for N. Since N was an arbitrary kG-submodule of V, it follows that V must be semisimple as a kG-module.