Problem 1 Scott Andrews, Erica Shannon

Let $\mathbb{F}$ be a field, and let A be a finite-dimensional $\mathbb{F}$-algebra. Let $\rho : A \to \mathrm{End}_{\mathbb{F}}(A)$ be the regular representation. Let $\langle x, y\rangle$ denote the trace form on A, such that

$$\langle x, y\rangle = \mathrm{Tr}(\rho(xy))$$

a. Show that if the trace form is nondegenerate then A is semisimple.

Let L be a nil left ideal of A, and let $x \in L$. Then there exists some k such that $x^k = 0$, thus $\rho(x)^k = 0$. It follows that $\rho(x)$ is a nilpotent transformation, hence has trace 0.

Let $x \in L$; then for all $y \in A$,

$$\langle x, y\rangle = \langle y, x\rangle = \mathrm{Tr}(\rho(yx)) = 0$$

as $yx \in L$. If the trace form is nondegenerate, this implies that $x = 0$, hence A contains no nonzero nil left ideals. As $\mathrm{rad}(A)$ is the largest nil left ideal of A (A is left artinian), it follows that $\mathrm{rad}(A) = 0$. Hence A is Jacobson semisimple and left artinian, thus A is semisimple.

b. Prove that if the trace form is degenerate and $\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F}) > \dim_{\mathbb{F}}(A)$ then A is not semisimple.

We begin with a series of lemmas towards the desired result.

Lemma 1. Let $x \in M_n(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 or of characteristic greater than n. If $\mathrm{Tr}(x^k) = 0$ for all $0 < k \leq n$, then x has 0 as an eigenvalue.

Proof. Let $f(t) = a_0 + a_1 t + ... + t^n$ be the characteristic polynomial of x; then $f(x) = 0$. It follows that

$$\begin{aligned} 0 &= \mathrm{Tr}(f(x)) \\ &= \sum_{k=0}^{n} a_k \mathrm{Tr}(x^k) \\ &= a_0 \mathrm{Tr}(I) \end{aligned}$$

By the restriction on the characteristic of $\mathbb{F}$, $\mathrm{Tr}(I) \neq 0$, hence $a_0 = 0$. Thus $f(t)$ has t as a factor, and 0 is an eigenvalue of x.

□

Lemma 2. Let $n > 0$ and let $\mathbb{F}$ be a field of characteristic 0 or of characteristic greater than n. Then the only solution to the system of equations

$$\begin{aligned} x_1 + x_2 + ... + x_n &= 0 \\ x_1^2 + x_2^2 + ... + x_n^2 &= 0 \\ \vdots \quad \vdots \quad &\vdots \\ x_1^n + x_2^n + ... + x_n^n &= 0 \end{aligned}$$

is $x_1 = x_2 = ... = x_n = 0$.

Proof. We will proceed by induction on n; the claim is true if $n = 1$. Assume that the claim is true for $n-1$, and that the characteristic of $\mathbb{F}$ is 0 or less than n. Define the matrix $x = \mathrm{diag}(x_1, x_2, ..., x_n)$. The above system of equations is equivalent to the statement that $\mathrm{Tr}(x^k) = 0$ for all $0 < k \le n$. By lemma 1, x has 0 as an eigenvalue, hence $x_i = 0$ for some i. Without loss of generality, assume $x_n = 0$. We can conclude from the system of equations that

$$\begin{aligned} x_1 + x_2 + ... + x_{n-1} &= 0 \\ x_1^2 + x_2^2 + ... + x_{n-1}^2 &= 0 \\ \vdots \quad \vdots \quad &\vdots \\ x_1^{n-1} + x_2^{n-1} + ... + x_{n-1}^{n-1} &= 0 \end{aligned}$$

By the inductive hypothesis, $x_i = 0$ for all i.

$\square$

We now return to the proof of main statement. Let L be the subset of A defined by $L = \{x \in A \mid \langle x, y \rangle = 0 \text{ for all } y \in A\}$.

Claim. L is a left ideal of A.

Proof. Let $x \in L$, and let $y, z \in A$. Then

$$\begin{aligned} \langle yx, z \rangle &= \mathrm{Tr}(\rho(yx)\rho(z)) \\ &= \mathrm{Tr}(\rho(z)\rho(yx)) \\ &= \mathrm{Tr}(\rho(zy)\rho(x)) \\ &= \langle zy, x \rangle \\ &= 0 \end{aligned}$$

hence $yx \in L$.

$\square$

Assume that the trace form is degenerate and $\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F}) > \dim_{\mathbb{F}}(A)$.

Claim. Let $x \in L$; then x is nilpotent.

Proof. As $x \in L$, $x^k \in L$ for all $0 < k \le n$. It follows that

$$\mathrm{Tr}(\rho(x^k)) = \langle x^k, 1 \rangle = 0$$

Let $\{x_i \mid 1 \le i \le n\}$ be a complete set of eigenvalues of $\rho(x)$; then

$$\begin{aligned} x_1 + x_2 + ... + x_n &= 0 \\ x_1^2 + x_2^2 + ... + x_n^2 &= 0 \\ \vdots \quad \vdots \quad &\vdots \\ x_1^n + x_2^n + ... + x_n^n &= 0 \end{aligned}$$

By lemma 2, $x_i = 0$ for all i, hence $\rho(x)$ is nilpotent. Let k be such that $\rho(x)^k = 0$; then $\rho(x^k) = 0$. As ρ is a faithful representation, $x^k = 0$, and x is nilpotent.

$\square$

It follows that L is a nil left ideal of A, and as the trace form is degenerate, L is nonempty. As $\mathrm{rad}(A)$ is the largest nil left ideal of A (A is left artinian), it follows that $\mathrm{rad}(A) \neq 0$. Thus A is not semisimple.

c. Maschke's Theorem: If $\mathbb{F}$ is a field, G is a finite group, and $\mathrm{char}(\mathbb{F})$ does not divide $|G|$, then $\mathbb{F}[G]$ is semisimple.

Proof. Let $\{g \mid g \in G\}$ be the standard basis of $\mathbb{F}[G]$, and let B be the matrix of the trace form relative to some ordering of the basis. Note that

$$\mathrm{Tr}(\rho(gh)) = \begin{cases} |G| & \text{if } h = g^{-1} \\ 0 & \text{otherwise} \end{cases}$$

As the map $g \mapsto g^{-1}$ is a permutation of G, $B = |G|P$ for some permutation matrix P. P is invertible, hence B has inverse given by $B^{-1} = |G|^{-1}P^{-1}$. It follows that the trace form is nondegenerate, and $\mathbb{F}[G]$ is semisimple.

□

d. $M_n(\mathbb{F})$ is semisimple if $\mathbb{F}$ has characteristic not dividing n.

Proof. Let e_{ij} denote the matrix in $M_n(\mathbb{F})$ with a 1 in the ijth entry and 0's elsewhere. The set $\{e_{ij} \mid 1 \leq i, j \leq n\}$ forms a basis of $M_n(\mathbb{F})$; let B be the matrix of the trace form relative to some ordering of this basis. Note that $e_{ij}e_{kl} = \delta_{jk}e_{il}$, hence $\rho(e_{ij})$ fixes no basis elements if $i \neq j$ and $\rho(e_{ii})$ fixes n basis elements. It follows that $\mathrm{Tr}(\rho(e_{ij})) = \delta_{ij}n$, and $\mathrm{Tr}(\rho(e_{ij}e_{kl})) = \delta_{il}\delta_{jk}n$. In particular, each row and column of B has one entry which is n and the rest of the entries are 0's. Thus $B = nP$ for some permutation matrix P, and B is invertible with inverse $n^{-1}P^{-1}$. It follows that the trace form is nondegenerate, and $M_n(\mathbb{F})$ is semisimple.

□