Problem 5.6
Homework 3

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Lemma. *The element* $\sum_{i=0}^{\infty} a_i t^i \in R[[t]]$ *has a left inverse* $\sum_{i=0}^{\infty} b_i t^i \in R[[t]]$ *if and only if* a_0 *has a left inverse in* R.

Proof. We have the product

$$\sum_{i=0}^{\infty} b_i t^i \sum_{i=0}^{\infty} a_i t^i = \sum_{i=1}^{\infty} \sum_{j=0}^{i} (b_{i-j} a_j) t^i.$$

If $\sum_{i=0}^{\infty} b_i t^i \sum_{i=0}^{\infty} a_i t^i = 1$ then clearly $b_0 a_0 = 1$ so a_0 has a left inverse.

Conversely, we will prove by induction on i that there exists a set of $b_i \in R$ such that $b_0 a_0 = 1$ and $\sum_{j=0}^{i} (b_{i-j} a_j) = 0$ for $i \geq 1$. Clearly b_0 exists by assumption. Assume the claim is true for $i-1$. Define $b_i = -\sum_{j=1}^{i} (b_{i-j} a_j) b_0$. Then

$$\sum_{j=0}^{i} (b_{i-j} a_j) = b_i a_0 + \sum_{j=1}^{i} (b_{i-j} a_j) = -\sum_{j=1}^{i} (b_{i-j} a_j) + \sum_{j=1}^{i} (b_{i-j} a_j) = 0$$

so the claim holds for i, hence for all i. This sequence of b_i clearly generates a left inverse of $\sum_{i=0}^{\infty} a_i t^i$. □

Problem 5.6. *Let* J *be the Jacobson radical of ring* R. *Then the Jacobson radical* $\mathfrak{R}$ *of* $R[[t]]$ *is* $J + tR[[t]]$.

Proof. Let $\sum_{i=0}^{\infty} a_i t^i \in \mathfrak{R}$. Then $1 + b \sum_{i=0}^{\infty} a_i t^i$ has a left inverse in $R[[t]]$ for any $b \in R$, hence $1 + b a_0$ has a left inverse in R for all $b \in R$. This is exactly the condition for $a_0 \in J$, hence $\mathfrak{R} \subseteq J + tR[[t]]$.

Conversely, if $\sum_{i=0}^{\infty} a_i t^i \in J + tR[[t]]$, ie $a_0 \in J$ and $a_i \in R$ arbitrary for $i \geq 1$, then

$$1 + \sum_{i=0}^{\infty} b_i t^i \sum_{i=0}^{\infty} a_i t^i = (1 + b_0 a_0) + \sum_{i=1}^{\infty} \sum_{j=0}^{i} (b_{i-j} a_j) t^i$$

and $1 + b_0 a_0$ has a left inverse in R, hence $1 + \sum_{i=0}^{\infty} b_i t^i \sum_{i=0}^{\infty} a_i t^i$ has a left inverse in $R[[t]]$, with $\sum_{i=0}^{\infty} b_i t^i$ arbitrary. Hence $\sum_{i=0}^{\infty} a_i t^i \in \mathfrak{R}$ and $J + tR[[t]] \subseteq \mathfrak{R}$.

□