Theory of Rings Homework

M. Hartman, A. Moore

Rings3p4.8:

An ideal $I \subsetneq R$ is called a maximal ideal of R if there is no ideal of R strictly between I and R. Show that any maximal ideal I of R is the annihilator of some simple left R-module, but not conversely. Defining $\mathrm{rad}'(R)$ to be the intersection of all maximal ideals of R, show that $\mathrm{rad}(R) \subseteq \mathrm{rad}'(R)$, and give an example to show that this may be a strict inclusion. ($\mathrm{rad}'(R)$ is called the *Brown-McCoy radical* of R.)

Proof: Let I be a maximal ideal in R. We can create a simple left R-module R/I_l where I_l is a maximal left ideal such that $I \subseteq I_l$. Then $i \cdot R/I_l = 0$ for all $i \in I$. Thus, $I \subseteq ann(R/I_l)$. $ann(R/I_l)$ is a proper ideal of R, since $1 \notin ann(R/I_l)$. But by the maximality of I and $I \subseteq ann(R/I_l) \subsetneq R$, we have $I = ann(R/I_l)$. Thus, every maximal ideal is the annihilator of some simple R-module.

To disprove the converse, we need to find a simple R-module whose annihilator is not a maximal ideal. Consider Problem 1 from homework 2. Let $R = \mathrm{E}nd_D(V)$ where V is an infinite dimensional D-vector space. Then V is a simple left R-module, and the annilator of V is (0). But (0) is not a maximal ideal in R.

To show that $\mathrm{rad}(R) \subseteq \mathrm{rad}'(R)$, by Corollary 4.2, we have $\mathrm{rad}(R) = \cap ann(M)$ where M ranges over all the simple left R-modules. Choose a maximal ideal I_0, and let M_0 be a simple module whose annihilator is I_0. The existence of M_0 is guaranteed by the argument of the first paragraph. Then $\mathrm{rad}(R) = \cap\, ann(M) \subseteq ann(M_0) = I_0$. But this is true for all maximal ideals I. Thus, $\mathrm{rad}(R) \subseteq \cap\, I = \mathrm{rad}'(R)$.

To show the inclusion can be strict, it has been shown that ideals of $R = \mathrm{E}nd_D(V)$ where V is an infinite dimensional D-vector space are of two types; $I_{<\lambda}$ (containing all elements of rank less than λ) and $I_{\leq\lambda}$. In particular, there is a total ordering in ideals. Thus, there will exist a unique maximal ideal containing all elements of rank less than $\mathrm{D}im_D(V)$. Thus, the Brown-McCoy radical will be $I_{<\lambda}$ where $\lambda = \mathrm{D}im_D(V)$, and thus $I_{<\lambda} \neq (0)$. However, there is a simple left R module, namely V, whose annihilator is (0). Thus, the Jacobson Radical is $(0) \neq I_{<\lambda}$ which is the Brown-McCoy radical.