MATH 6250: Theory of Rings
Homework 3
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Let R be a ring possibly without an identity. In R, define $a \circ b = a + b - ab$. It can be shown that this binary operation is associative, and that $(R, \circ)$ is a monoid with zero as the identity element. An element $a \in R$ is called *left* (resp. *right*) *quasi-regular* if a has a left (resp. right) inverse in the monoid $(R, \circ)$ with identity. If a is both left and right quasi-regular, we say that a is *quasi-regular*. A set $I \subseteq R$ is called quasi-regular (resp. left or right quasi-regular) if every element of I is quasi-regular (resp. left or right quasi-regular).

Ex. 4.4 Define the Jacobson radical of R by

$$rad\, R = \{a \in R : Ra \text{ is left quasi-regular}\}.$$

Show that $rad\, R$ is a quasi-regular ideal which contains every quasi-regular left (resp. right) ideal of R. (In particular, $rad\, R$ contains every nil left or right ideal of R.) Show that, if R has an identity, the definition of $rad\, R$ here agrees with the one given in the introduction to this section.

Solution.

Claim 1. If R has an identity 1, the map $\varphi : (R, \circ) \to (R, \times)$ sending a to $1 - a$ is a monoid isomorphism. In this case, an element a is left (right) quasi-regular if and only if $1 - a$ has a left (right) inverse with respect to ring multiplication.

Proof. The map is obviously a bijection with inverse $a \mapsto 1 - a$. To see that it is a homomorphism: $\varphi(a \circ b) = \varphi(a + b - ab) = 1 - a - b + ab = (1 - a)(1 - b) = \varphi(a)\varphi(b)$. The second part of the claim is trivial.

We can use this claim and the fact that in analysis we can write the inverse of $1 - a$ as an infinite geometric series to find quasi-inverses to elements in $(R, \circ)$, even when there is no identity in R.

Claim 2. If ab is left quasi-regular, then so is ba.

Proof. Let c be the left quasi-inverse of ab, i.e. $c \circ (ab) = 0$. Then using the previous claim we can find the left quasi-inverse of ba to be $bca - ba$. We have to check that it is indeed a left quasi-inverse: $(bca - ba) \circ (ba) = bca - ba + ba - bcaba + baba = bca - bcaba + baba = b(c - cab + ab)a = b(c \circ (ab))a = 0$.

Claim 3. Any nilpotent element is quasi-regular.

Proof. Let a be a nilpotent element, so $a^{2^k} = 0$ for some positive integer k. As we have $a^{2^k} = -a^{2^{k-1}} + a^{2^{k-1}} + a^{2^{k-1}}a^{2^{k-1}} = (-a^{2^{k-1}}) \circ a^{2^{k-1}}$, we can write recursively that $0 = a^{2^k} = (-a^{2^{k-1}}) \circ ((-a^{2^{k-2}}) \circ \cdots \circ ((-a) \circ a) \cdots)$. Because the operation $\circ$ is associative, we found a left quasi-inverse to a. We can find a right quasi-inverse by a similar method.

Claim 4. If a left ideal is left quasi-regular, then it is quasi-regular.

Proof. Let $I \subset R$ be a left quasi-regular left ideal, and $a \in I$. Then a is left quasi-regular, so $c \circ a = c + a - ca = 0$ for some $c \in R$. But then $c = ca - a \in I$, so c is also left quasi-regular and $d \circ c = 0$, for some $d \in R$. As $d = d \circ c \circ a = a$, we have that $a \circ c = 0$, and thus a is right quasi-regular and also quasi-regular. As a was an arbitrary element of I, I is quasi-regular.

Proof of 4.4 First we prove that $rad\, R$ is an ideal. Let $a, b \in rad\, R$. To prove that $a + b \in rad\, R$, we need to show that for any $r \in R$, $r(a + b) = ra + rb$ is left quasi-regular. Let c be the left quasi-inverse of ra which exists because $a \in rad\, R$. Then $c \circ (ra + rb) = c + ra + rb - cra - crb = rb - crb = (r - cr)b$. The element $(r - cr)b$ has a left quasi-inverse d, because $b \in rad\, R$, thus $d \circ c$ is a left quasi-inverse of $ra + rb$ and $a + b \in rad\, R$. Now let $s \in R$ be arbitrary, we need to show that $sa, as \in rad\, R$, i.e. Rsa and Ras are left quasi-regular. Rsa is left quasi-regular, because Ra is left quasi-regular. Any element ras is left quasi-regular using Claim 2 and the fact that sra is left quasi-regular. So Ras is also left quasi-regular and $rad\, R$ is an ideal.

Next we prove that $rad\, R$ is quasi-regular. As $rad\, R$ is an ideal, thus a left ideal, because of Claim 4, we only need to prove that $rad\, R$ is left quasi-regular. Let $a \in rad\, R$, thus a^2 has a left quasi-inverse c. Then $c + ca - a$ is a left quasi-inverse to a: $(c + ca - a) \circ a = c + ca - a + a - ca - ca^2 + a^2 = c + a^2 - ca^2 = c \circ (a^2) = 0$. Thus $rad\, R$ is quasi-regular.

Next we prove that if I is a quasi-regular left ideal then $I \subset rad\, R$. Let $a \in I$. Then $Ra \subset I$ as I is a left ideal, and so Ra is quasi-regular. Thus $a \in rad\, R$ and $I \subset rad\, R$.

Now let I be a quasi-regular right ideal. We need to prove that $I \subset rad\, R$. Let $a \in I$, then $aR \subset I$, so for any $r \in R$, ar is quasi-regular. But then using Claim 2 we have that ra is left quasi-regular, so $a \in rad\, R$ and $I \subset rad\, R$. Thus indeed $rad\, R$ contains every quasi-regular left or right ideal of R.

Claim 3 shows that a nil left or right ideal is quasi-regular, thus indeed $rad\, R$ contains every nil left or right ideal of R.

It remains to show that if R has identity, then the two definitions of $rad\, R$ are equivalent. Let J be the intersection of all maximal left ideals of R. Then by Lemma 4.1, $a \in J$ is equivalent to the statement that $1 - ra$ is left-invertible (with respect to the multiplication of R) for any $r \in R$. By Claim 1 this is equivalent to ra being left quasi-regular for any $r \in R$. But according to the definition of $rad\, R$ this is equivalent to a being an element of $rad\, R$. Thus the two definitions of the Jacobson radical are indeed equivalent.