Problem. For any ring R, let $GL_n(R)$ denote the group of units of $\mathbb{M}_n(R)$. Show that for any ideal $I \subseteq rad\, R$, the natural map $GL_n(R) \to GL_n(R/I)$ is surjective. (**Hint.** First prove this for $n = 1$.)

Solution. Let f denote the natural map $\mathbb{M}_n(R) \to \mathbb{M}_n(R/I)$. Suppose $u \in \mathbb{M}_n(R)$ with $f(u)$ a unit in $\mathbb{M}_n(R/I)$. Then $f(u)$ is invertible in $\mathbb{M}_n(R/I)$; in other words, there exists $v \in \mathbb{M}_n(R)$ such that $uv = 1 + r$ where $r \in \mathbb{M}_n(I)$. Since $I \subseteq rad\, R$ we have also that $r \in \mathbb{M}_n(rad\, R)$.

We proved in non-Lam Problem 3 that $rad\,(\mathbb{M}_n(R)) = \mathbb{M}_n(rad\, R)$, so $r \in rad\,(\mathbb{M}_n(R))$. So by characterization of the radical, $1 + r = uv$ is a unit in $\mathbb{M}_n(R)$. Hence there is a matrix w such that $uvw = 1$. So u has a right inverse in $\mathbb{M}_n(R)$.

The same argument shows that u has a left inverse in $\mathbb{M}_n(R)$, and therefore u is in $GL_n(R)$. So any matrix in $f^{-1}(GL_n(R/I))$ is in $GL_n(R)$. Since we know that f is surjective, it follows that $f|_{GL_n(R)}$ must be a surjection onto $GL_n(R/I)$.