rings3p3

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(a) Let R be a ring and $e \in R$ an idempotent. Show that $\text{rad}(eRe) = e(\text{rad}(R))e$.

Proof. Note that e is the multiplicative identity in eRe. Let $r \in \text{rad}(eRe)$, then $e - xr$ is left invertible for any $x \in eRe$. Let $y \in R$, then there is some $b \in eRe$ such that $b(e - eye \cdot r) = e$, but since b and r are in eRe and $e^2 = e$, we have $b(1 - yr) = e$. Now

$$(1 + yrb)(1 - yr) = (1 - yr) + yrb(1 - yr) = 1 - yr + yre = 1 - yr + yr = 1,$$

so $1 - yr$ has a left inverse in R. Since y was arbitrary, $r \in \text{rad}(R)$. Now we have

$$\text{rad}(eRe) \subseteq \text{rad}(R).$$

Also, since $r \in \text{rad}(eRe)$, $r \in eRe$, and so $r = eae$ for some $a \in R$. This gives us

$$ere = e^2ae^2 = eae = r,$$

and since $r \in \text{rad}(eRe) \subseteq \text{rad}(R)$, $r \in e(\text{rad}(R))e$, so

$$\text{rad}(eRe) \subseteq e(\text{rad}(R))e. \tag{1}$$

Now let $r \in e(\text{rad}(R))e$, $r \in (\text{rad}(R))$, so for all $y' \in R$, there is an $x \in R$ such that $x(1 - y'r) = 1$. This gives us that for all $y \in eRe$,

$$e = e^2 = ex(1 - yr)e = ex(e - yr) = exe(e - yr),$$

hence for all $y \in eRe$, $e - yr$ has a left inverse in eRe, and therefore $r \in \text{rad}(eRe)$ and now

$$e(\text{rad}(R))e \subseteq \text{rad}(eRe). \tag{2}$$

With (1) and (2) we have the desired result.

□

(b) Show that $\text{rad}(M_n(R)) = M_n(\text{rad}(R))$.

Proof. Note that since the radical of a ring is an ideal in that ring, and ideals of matrix rings over a ring R are matrix rings over an ideal of R. Given this, let

$$\text{rad}(M_n(R)) = M_n(U)$$

where U is some ideal in R. Let $e = E_{1,1}$ be the idempotent unit matrix with 1 in the first position. By part (a) we have that

$$\text{rad}(eRe) = e(\text{rad}(R))e.$$

Combining these facts we have

$$eM_n(U)e = \text{rad}(eM_n(R)e).$$

The RHS is the set of matrices with an element from R in the first position that are also elements of the radical of $eM_n(R)e$. This implies that the element in the first position must be an element of the radical of R. The LHS is the set of matrices with an element from U in the first position. Since these two sets are equal

$$U = \mathrm{rad}(R),$$

and we have the desired result.

$\square$