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Show that any left noetherian ring has a nilpotent ideal which contains every left or right nil ideal.

Let R be a left noetherian ring.

Claim 1: The ideal generated by any left or right nilpotent ideal is nilpotent.

Proof. Let I be a nilpotent left ideal of R with $I^k = 0$. Then

$$\begin{aligned}(IR)^k &= I(RI)^{k-1}R \\ &= I(I^{k-1})R \\ &= 0\end{aligned}$$

hence IR is a nilpotent (two-sided) ideal of R . The proof is analogous for a right ideal. □

Claim 2: For $a \in R$, aR is nil if and only if Ra is nil.

Proof. Let aR be nil; then for each $r \in R$, there exists k such that $(ar)^k = 0$. It follows that

$$\begin{aligned}(ra)^{k+1} &= r(ar)^k a \\ &= 0\end{aligned}$$

and Ra is nil. The other direction is analogous. □

Claim 3: Let aR be nil, and let $b \in aR$ be a nonzero element such that $\text{ann}_l(b)$ is maximal among left annihilators of nonzero elements. Then $(bR)^2 = 0$.

Proof. First observe that such an element b exists as R is left noetherian. Let $r \in R$; then

$$\text{ann}_l(b) \subseteq \text{ann}_l(br)$$

As $\text{ann}_l(b)$ is maximal among left annihilators of nonzero elements, either $br = 0$ or $\text{ann}_l(b) = \text{ann}_l(br)$.

Let $r \in R$; as aR is nil, there exists minimal k such that $(br)^k = 0$. If $k = 1$, then $br = 0$ and $brb = 0$. If $k > 1$, then $(br)^{k-1} \neq 0$ and $\text{ann}_l(b) = \text{ann}_l((br)^{k-1})$. As $br \in \text{ann}_l((br)^{k-1})$, it follows that $brb = 0$. □

If R contains a nonzero nil right (or left) ideal I , then for a nonzero element $a \in I$ the ideal aR (or Ra) is nil. By claim 2, in either case aR is a nil right ideal. By claim 3 there exists a nonzero $b \in aR$ such that bR is a nilpotent. Finally, by claim 1 the ideal RbR is nilpotent.

If R contains no nonzero nil left or right ideals, the result is trivial. Assume R has some nonzero nil left or right ideal; then R contains a nonzero nilpotent ideal. Let I be a maximal nilpotent ideal of R (as R is left noetherian such an ideal exists). Assume R/I has some nonzero nilpotent ideal J/I ; then $J^k \subseteq I$ for some k , and J is a nilpotent ideal of R properly containing I . This is a contradiction, so no such J exists, hence R/I contains no nonzero nil left or right ideals. If K is a nil left (or right) ideal of R , then $(K + I)/I$ is a nil left (or right) ideal of R/I . It follows that $K + I = I$, and $K \subseteq I$.